FILTRATION ASSOCIATED TO TORSION SEMI-STABLE REPRESENTATIONS

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Abstract. Let p be an odd prime, K a finite extension of $\mathbb{Q}_p$ and $G := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ the Galois group. We construct and study filtration structures associated torsion semi-stable representations of G. In particular, we prove that two semi-stable representations share the same p-adic Hodge type if they are congruent modulo p^n with $n \geq c'$, where c' is a constant only depending on K and the differences between the maximal and minimal Hodge-Tate weights of two representations. As an application, we reprove a part of Kisin's result: the existence of a quotient of the universal Galois deformation ring which parameterizes semi-stable representations with a fixed p-adic Hodge type.

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Contents

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1. Introduction

Let k be a perfect field of characteristic $p \geq 3$, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[1/p]$, K/K_0 a finite totally ramified extension, $G := \mathrm{Gal}(\overline{K}/K)$. The aim of this paper to study filtration structure attached to torsion semi-stable representations.

If V is a semi-stable representation of G then V can be naturally attached to filtration structure because semi-stable representations are classified by filtered (φ, N)-modules via classical p-adic Hodge theory. Since V always admits integral structures and torsion structures, i.e., G-stable $\mathbb{Z}_p$-lattices and torsion representation obtained by quotients of such lattices, it is natural to ask if we can associate similar filtration to those integral and torsion structures. If K is unramified and V is crystalline with Hodge-Tate weights in $\{0, \dots, p-2\}$ then one can attach such integral and torsion structure via Fontaine-Laffaille theory [FL82]. The aim of this paper is to construct and study such structures in a more general setting, in particular, without restriction of ramification and Hodge-Tate weights.

More precisely, fix an integer $r \geq 0$. Let $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Q}_p}$ denote the category of semi-stable representations of G with Hodge-Tate weights in $\{0, \dots, r\}$, $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p}$ denote the category of G-stable $\mathbb{Z}_p$-lattices inside representations which are objects in $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Q}_p}$ and $\mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor}}$ denote the category of p-power torsion representations T such that there exist lattices $\Lambda_1, \Lambda_2 \in \mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p}$ satisfying $\Lambda_1 \subset \Lambda_2$ and $T \simeq \Lambda_2/\Lambda_1$. The objects in $\mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor}}$ are also called *torsion semi-stable representations with Hodge-Tate weights in* $\{0, \dots, r\}$. In [Liu11], for any $\Lambda \in \mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p}$ with $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda$, one can construct a $W(k)$-lattice $M_{\mathrm{st}}(\Lambda) \subset D_{\mathrm{st}}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^\vee)^G$ which is φ-stable and N-stable, where $V^\vee$ is the $\mathbb{Q}_p$-dual of V [2]. Note that $D_K := K \otimes_{K_0} D_{\mathrm{st}}(V)$ has a natural filtration structure $\mathrm{Fil}^i D_K$ induced from $(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^\vee)^G$. Now set $M_K := \mathcal{O}_K \otimes_{W(k)} M_{\mathrm{st}}(\Lambda)$. It is natural to define that $\mathrm{Fil}^i M_K := M_K \cap \mathrm{Fil}^i D_K$. For any $T \in \mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor}}$, let $j : \Lambda_1 \subset \Lambda_2 \in \mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p}$ be the inclusion of two lattices such that $T \simeq \Lambda_2/\Lambda_1$. Since M_{st} is a contravariant functor, there exists a $W(k)$-linear map $M_{\mathrm{st}}(j) : M_{\mathrm{st}}(\Lambda_2) \to M_{\mathrm{st}}(\Lambda_1)$. In fact, $M_{\mathrm{st}}(j)$ is an injection (see Corollary 3.2.4 in [Liu11]). So we define $M_{\mathrm{st},j}(T) := M_{\mathrm{st}}(\Lambda_1)/M_{\mathrm{st}}(\Lambda_2)$ and associate a filtration structure on $M_{\mathrm{st},j}(T)_K := \mathcal{O}_K \otimes_{W(k)} M_{\mathrm{st},j}(T)$ via $\mathrm{Fil}^i M_{\mathrm{st},j}(T)_K = q_K(\mathrm{Fil}^i M_{\mathrm{st}}(\Lambda_1)_K)$, where q_K is the natural projection $q_K : M_{\mathrm{st}}(\Lambda_1)_K \twoheadrightarrow M_{\mathrm{st},j}(T)_K$. Note the above construction does depend on the choice of pair of lattices $j : \Lambda_1 \subset \Lambda_2$ such that $T \simeq \Lambda_2/\Lambda_1$. However we prove that there exists a constant c only depending on r and K such that the construction of $\mathrm{Fil}^i M_{\mathrm{st},j}(T)_K$ is "independent on" the choice of j up to a p^c-power (see Theorem 2.1.3 for the precise statement).

In the last section, we use our theory to understand p-adic Hodge type. It turns out that the p-adic Hodge type can be read from $p^{c'}$-torsion level of the representation with c' a constant only depending on K and r. More precisely, we proved the following theorem.

Theorem 1.0.1. *Let E be a finite extension of $\mathbb{Q}_p$ and $\rho_i : G \to \mathrm{GL}_d(\mathcal{O}_E)$ for $i = 1, 2$ two Galois representations such that $V_i := E \otimes_{\mathcal{O}_E} \rho_i$ is semi-stable with Hodge-Tate weights in $\{0, \dots, r\}$. There exists a constant c' only depending on K*

[2]Our convention is always slightly different from the traditional convention up to $*$, see Convention 2.1.1 for details.

and r such that if $\rho_1 \equiv \rho_2 \mod p^n$ with $n \geq c'$ then V_1 and V_2 has the same p-adic Hodge type.

In fact, we proved a more general result Theorem 4.2.8, which allows us to recover a part of the main theorem in [Kis08]. Let E be a finite extension of $\mathbb{Q}_p$ with the residue field $\mathbb{F}$, $V_{\mathbb{F}} : G \to \mathrm{GL}_d(\mathbb{F})$ the Galois representation such that the universal deformation ring $R_{V_{\mathbb{F}}}$ of $V_{\mathbb{F}}$ exists. It turns out that $R_{V_{\mathbb{F}}}$ is a complete noetherian local $\mathcal{O}_E$-algebra and any ring homomorphism $x : R_{V_{\mathbb{F}}} \to A$ with A an $\mathcal{O}_E$-algebra defines a Galois representation $x : G \to \mathrm{GL}_d(A)$.

Theorem 1.0.2. *Fix a p-adic Hodge type $\mathbf{v}$. There exists a quotient $R^{\mathbf{v}}_{V_{\mathbb{F}}}$ of $R_{V_{\mathbb{F}}}$ such that for a finite E-algebra B, a map $x : R_{V_{\mathbb{F}}}[\frac{1}{p}] \to B$ factors though $R^{\mathbf{v}}_{V_{\mathbb{F}}}$ if and only if x is semi-stable with p-adic Hodge type $\mathbf{v}$.*

We remark that our construction is different from that of Kisin: We construct a sub-functor $D^{\mathbf{v}}$ of the Galois deformation functor D whose deformation admits a lift which is a semi-stable Galois representation with the p-adic Hodge type $\mathbf{v}$. We prove that $D^{\mathbf{v}}$ is pro-representable by $R^{\mathbf{v}}_{V_{\mathbb{F}}}$ if D is pro-representable by $R_{V_{\mathbb{F}}}$ and then the above theorem follows Theorem 4.2.8. It seems that we can fully recover Kisin's result at least for $p > 2$ if we also consider Galois type in $D^{\mathbf{v}}$. But we decide not to consider the refined result because we do not see any further advantage (except it looks more natural) of our construction comparing with that of Kisin.

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2. Filtration encoded in p-adic Hodge data

2.1. **Preliminary and definitions.** Recall k is a perfect field of characteristic $p > 2$, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[\frac{1}{p}]$, K/K_0 a finite totally ramified extension with degree e and $G := \mathrm{Gal}(\overline{K}/K)$. Throughout this paper, we fix a uniformiser $\pi \in K$ with the Eisenstein polynomial $E(u) \in W(k)[u]$ and a non-negative integer $r \geq 0$.

We denote by $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Q}_p}$ the category of semi-stable representations of G whose Hodge-Tate weights are in $\{0, \dots, r\}$, and by $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p}$ the category of G-stable $\mathbb{Z}_p$-lattices in representations which are in $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Q}_p}$. By definition, a *filtered (φ, N)-module* is a $W(k)$-module M endowed with:

- a $\varphi_{W(k)}$-semilinear map: $\varphi : M \to M$;
- a $W(k)$-linear map $N : M \to M$ such that $N\varphi = p\varphi N$;
- a decreasing filtration $(\mathrm{Fil}^i M_K)_{i\in\mathbb{Z}}$ on $M_K := \mathcal{O}_K \otimes_{W(k)} M$ by $\mathcal{O}_K$-submodules such that $\mathrm{Fil}^i M_K = M_K$ for $i \ll 0$ and $\mathrm{Fil}^i M_K = \{0\}$ for $i \gg 0$.

This definition is slightly different from that traditionally used in [Fon94b] because we need treat torsion representations. Morphisms between filtered (φ, N)-modules are $W(k)$-linear maps preserving all structures. We denote by $\mathrm{M}(\varphi, N, \mathrm{Fil})$ the category of filtered (φ, N)-modules. A *filtered (φ, N)-module over K_0* is a filtered (φ, N)-module D such that

- D is a finite dimensional K_0-vector space;
- φ_D is an injection (hence a bijection);
- $\mathrm{Fil}^i D_K$ are K-vector subspaces of $D_K := K \otimes_{K_0} D$.

By [CF00] and [Fon94a], the functor [3]

$$D^*_{\mathrm{st}}(V): \quad V \mapsto (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^G$$

induces an equivalence between the category $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Q}_p}$ and the category of *weakly admissible* filtered (φ, N)-modules over K_0 satisfying $\mathrm{Fil}^{-(r+1)} D_K = D_K$ and $\mathrm{Fil}^0 D_K = 0$. See [CF00] for the definition of weakly admissibility.

In the sequel, we will instead use the contravariant functor $D_{\mathrm{st}}(V) := D^*_{\mathrm{st}}(V^\vee)$, where $V^\vee$ is the dual representation of V, because contravariant functors are more convenient in the integral theory. So let us remind the readers the problem of notations.

Convention 2.1.1. Here we use slightly different conventions from those in [CF00], where D_{st} defined here is denoted by D^*_{st}. Since *contravariant* functors instead of covariant functors dominate this paper, use D_{st} to denote the contravariant functor will be more convenient. For any finite $\mathbb{Z}_p$-module ($\mathbb{Q}_p$-module) V, we use $V^\vee$ to denote its $\mathbb{Z}_p$-dual ($\mathbb{Q}_p$-dual). In particular, if V is killed by some p-power, $V^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(V, \mathbb{Q}_p/\mathbb{Z}_p)$. We will define p-adic Hodge structures such as Frobenius, monodromy on many different rings and modules. To distinguish them, we sometime add subscripts to indicate where those structures are defined. For example, $\varphi_{\mathfrak{M}}$ is the Frobenius defined on $\mathfrak{M}$. We always drop these subscripts if no confusions arise. Throughout this paper, we reserve φ and N for various types of Frobenius and monodromy respectively. We denote $\gamma_i(x)$, $\mathrm{M}_d(A)$ and Id for the standard divided power $\frac{x^i}{i!}$, the ring of $d \times d$-matrices with coefficients in ring A and the identity map respectively. Let A be a finite $\mathbb{Z}_p$-algebra and M an A-module. We always denote $M_K := \mathcal{O}_K \otimes_{\mathbb{Z}_p} M$, which is an $A_K := \mathcal{O}_K \otimes_{\mathbb{Z}_p} A$-module.

Let D be a filtered (φ, N)-module over K_0. Following [Liu11], a *lattice* M in D is a $W(k)$-submodule M of D such that

- M is $W(k)$-finite free and $M[\frac{1}{p}] \overset{\sim}{\to} D$
- M is stable under φ, N, i.e., $\varphi(M) \subset M$, $N(M) \subset M$.

There is a natural filtration structure on $M_K := \mathcal{O}_K \otimes_{W(k)} M$ defined by $\mathrm{Fil}^i M_K := M_K \cap \mathrm{Fil}^i D_K$. Hence M is an object in $\mathrm{M}(\varphi, N, \mathrm{Fil})$. We use $\mathrm{L}^r(\varphi, N, \mathrm{Fil})$ to denote the full subcategory of $\mathrm{M}(\varphi, N, \mathrm{Fil})$ whose objects are lattices in filtered (φ, N)-modules satisfying $\mathrm{Fil}^0 D_K = D_K$ and $\mathrm{Fil}^{r+1}_K D_K = 0$, and $\mathrm{M}^r(\varphi, N, \mathrm{Fil})$ to denote the fully subcategory of $\mathrm{M}(\varphi, N, \mathrm{Fil})$ whose objects are finite $W(k)$-modules M such that $\mathrm{Fil}^0 M_K = M_K$ and $\mathrm{Fil}^{r+1} M_K = 0$. Apparently, $\mathrm{L}^r(\varphi, N, \mathrm{Fil})$ is a full subcategory of $\mathrm{M}^r(\varphi, N, \mathrm{Fil})$. Let $\mathrm{M}^r_{\mathrm{tor}}(\varphi, N, \mathrm{Fil})$ denote the full category of $\mathrm{M}^r(\varphi, N, \mathrm{Fil})$ whose objects is killed by some p-power. For any $\mathcal{O}_K$-module N with decreasing filtration $\mathrm{Fil}^i N$, we define the graded module $\mathrm{gr}^i N := \mathrm{Fil}^i N / \mathrm{Fil}^{i+1} N$.

Now recall Theorem 2.1.3 in [Liu11], we have

Theorem 2.1.2. *There exists a left exact and faithful functor M_{st} from the category $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p}$ to the category $\mathrm{L}^r(\varphi, N, \mathrm{Fil})$. Moreover, let $M_{\mathrm{st}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ denote the functor*

[3] Note our notations are slightly different from those in [CF00].

M_{st} associated to the isogeny categories. Then there is a natural isomorphism between $M_{\mathrm{st}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and D_{st}. If $er < p-1$ then M_{st} is exact and fully faithful.

Now let us construct a torsion version of M_{st} via the above theorem as in §3 in [Liu11]. We denote $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{st},r}$ the category whose objects are *torsion semi-stable representations with Hodge-Tate weights in* $\{0,\dots,r\}$, in the sense that, for any $T \in \mathrm{Rep}_{\mathrm{tor}}^{\mathrm{st},r}$, there exist G-stable $\mathbb{Z}_p$-lattices $\Lambda \subset \Lambda'$ in a $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st},r}$ such that $T \simeq \Lambda'/\Lambda$ as $\mathbb{Z}_p[G]$-modules. We call the pair $\Lambda \subset \Lambda'$ a *lift* of T. Obviously, for any $T \in \mathrm{Rep}_{\mathrm{tor}}^{\mathrm{st},r}$, the lift is always not unique. A morphism between two lifts $j : L \subset L'$ (lifting T) and $\tilde{j} : \tilde{L} \subset \tilde{L}'$ (lifting $\tilde{T}$) is a morphism $\hat{f} : L' \to \tilde{L}'$ in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{st},r}$ such that $\hat{f}(L) \subset \tilde{L}$. $\hat{f}$ induces a morphism $f : T \to \tilde{T}$ in $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{st},r}$. We call $\hat{f}$ a *lift of* f (with respect to lifts j and $\tilde{j}$).

Let $j : \Lambda \hookrightarrow \Lambda'$ be a lift of T. By Theorem 2.1.2, we get a morphism $M_{\mathrm{st}}(j) : M_{\mathrm{st}}(\Lambda') \to M_{\mathrm{st}}(\Lambda)$ in $\mathrm{L}^r(\varphi, N, \mathrm{Fil})$. Corollary 3.2.4 in [Liu11] showed that $M_{\mathrm{st}}(j)$ is injective. Now write $\tilde{j} := M_{\mathrm{st}}(j)$ and set $M_{\mathrm{st},j}(T) := M_{\mathrm{st}}(\Lambda)/\tilde{j}(M_{\mathrm{st}}(\Lambda'))$. Then $M := M_{\mathrm{st},j}(T)$ has Frobenius φ and monodromy N induced from $M_{\mathrm{st}}(L)$. Now let us define filtration structure on $M_K := \mathcal{O}_K \otimes_{W(k)} M$. Let $q_K : M_{\mathrm{st}}(\Lambda)_K \to M_K$ be the natural projection where $M_{\mathrm{st}}(\Lambda)_K := \mathcal{O}_K \otimes_{W(k)} M_{\mathrm{st}}(\Lambda)$. We define

$$\mathrm{Fil}^i M_K := q_K(\mathrm{Fil}^i(M_{\mathrm{st}}(\Lambda)_K)) \subset M_K, \text{ for any } i \in \mathbb{Z}. \tag{2.1.1}$$

Now $M_{\mathrm{st},j}(T)$ is an object in $\mathrm{M}_{\mathrm{tor}}^r(\varphi, N, \mathrm{Fil})$. As usual, one can define $\mathrm{gr}^i M_K := \mathrm{Fil}^i M_K/\mathrm{Fil}^{i+1} M_K$. One can prove that $q_K(\mathrm{gr}^i(M_{\mathrm{st}}(\Lambda)_K)) = \mathrm{gr}^i M_K$ for any $i \in \mathbb{Z}$ (see Corollary 3.1.1). Now we can state one of the main results:

Theorem 2.1.3. *There exists a constant c only depending on $E(u)$ and r such that the following statement holds: for any morphism $f : T' \to T$ in $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{st},r}$ and any lift j', j of T', T respectively, there exists a morphism $\tilde{g} : M_{\mathrm{st},j}(T) \to M_{\mathrm{st},j'}(T')$ in $\mathrm{M}_{\mathrm{tor}}^r(\varphi, N, \mathrm{Fil})$ such that*

1. *if there exists a morphism of lifts $\hat{f} : j' \to j$ which lifts f then $\tilde{g} = p^c M_{\mathrm{st},\hat{f}}(f)$.*
2. *let $f' : T'' \to T'$ be a morphism in $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{st},r}$ with j'' the lift of T'' and $\tilde{g}' : M_{\mathrm{st},j'}(T') \to M_{\mathrm{st},j''}(T'')$ the morphism in $\mathrm{M}_{\mathrm{tor}}^r(\varphi, N, \mathrm{Fil})$ attached to f', j' and j''. If there exists a morphism of lifts $\hat{h} : j'' \to j$ which lifts $f \circ f'$ then $\tilde{g}' \circ \tilde{g} = p^{2c} M_{\mathrm{st},\hat{h}}(f \circ f')$.*

Corollary 2.1.4. *Notations as above, assume that $f : T' \to T$ is an isomorphism and $f' = f^{-1} : T \to T'$ is the inverse map. Then $\tilde{g}' \circ \tilde{g}|_{M_{\mathrm{st},j}(T)} = p^{2c}\mathrm{Id}|_{M_{\mathrm{st},j}(T)}$ and $\tilde{g} \circ \tilde{g}'|_{M_{\mathrm{st},j'}(T')} = p^{2c}\mathrm{Id}|_{M_{\mathrm{st},j'}(T')}$. Moreover, for any $i \in \mathbb{Z}$, $\tilde{g}' \circ \tilde{g}|_{\mathrm{gr}^i(M_{\mathrm{st},j}(T)_K)} = p^{2c}\mathrm{Id}|_{\mathrm{gr}^i(M_{\mathrm{st},j}(T)_K)}$ and $\tilde{g} \circ \tilde{g}'|_{\mathrm{gr}^i(M_{\mathrm{st},j'}(T')_K)} = p^{2c}\mathrm{Id}|_{\mathrm{gr}^i(M_{\mathrm{st},j'}(T')_K)}$.*

Remark 2.1.5. There are two differences between Theorem 3.1.1 in [Liu11] and the above theorem expect $p > 2$ here. First, the maps $\tilde{g}$ and $\tilde{g}'$ here not only preserve (φ, N)-structures (we ignore G_K-structures because we only discuss semi-stable representations here, not potentially semi-stable representations as in [Liu11]) but also filtration. In fact, we will see that $\tilde{g} = p^\alpha g$ and $\tilde{g}' = p^\alpha g'$ for g and g' in Theorem 3.1.1 in [Liu11] with α a constant only depending on $E(u)$ and r. That is, to preserve filtration, we need to multiply p^α to original g and g'. Consequently, the second difference, our constant c is always larger than $\mathfrak{c}$ in Theorem 3.1.1 in

[Liu11], and c is more complicated, depending on not only e and r but also $E(u)$ and r.

We use very similar strategy to prove the above theorem. The only difficulty is to deal with filtration which is much involved than other structures for torsion representations. Our main tool is Breuil modules. The next subsection is devoted to study various different filtration structures attached to Breuil modules.

2.2. **Filtration on Breuil modules.** Recall the fixed uniformiser $\pi \in K$ with Eisenstein polynomial $E(u)$. Put $\mathfrak{S} := W(k)[\![u]\!]$. $\mathfrak{S}$ is equipped with a Frobenius endomorphism φ via $u \mapsto u^p$ and the natural Frobenius on $W(k)$. A *φ-module* (over $\mathfrak{S}$) is an $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$. A morphism between two objects $(\mathfrak{M}_1, \varphi_1)$, $(\mathfrak{M}_2, \varphi_2)$ is an $\mathfrak{S}$-linear morphism compatible with the φ_i. We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. Write $S_{K_0} := S[\frac{1}{p}]$. There is a unique map (Frobenius) $\varphi : S \to S$ which extends the Frobenius on $\mathfrak{S}$. We write N_S for the $W(k)$-linear derivation on S such that $N_S(u) = -u$. Let $\mathrm{Fil}^i S$ denote the ideal which is the p-adic completion of the ideal generated by $\frac{E(u)^j}{j!}$ for $j \geq i$. Both $\mathfrak{S}$ and S can be regarded as subrings of $K_0[\![u]\!]$. Set $I_+ S = S \cap uK_0[\![u]\!]$.

Let $\mathcal{M}$ be an S-module of finite type and recall that r is a fixed nonnegative integer. In this subsection, filtration $\mathrm{Fil}^i\mathcal{M}$ of $\mathcal{M}$ for $i \in \mathbb{Z}$ are submodules of $\mathcal{M}$ satisfying the following *filtration conditions*:

- $\mathrm{Fil}^0\mathcal{M} = \mathcal{M}$ and $\mathrm{Fil}^{r+1}\mathcal{M} \subset \mathrm{Fil}^1 S\mathcal{M}$.
- $\mathrm{Fil}^{i+1}\mathcal{M} \subset \mathrm{Fil}^i\mathcal{M}$ and $\mathrm{Fil}^i S \mathrm{Fil}^j \mathcal{M} \subset \mathrm{Fil}^{i+j}\mathcal{M}$.

An operator N on $\mathcal{M}$ is called a *monodromy operator* on $\mathcal{M}$ if N is a $W(k)$-linear map $N : \mathcal{M} \to \mathcal{M}$ satisfying $N(sx) = N_S(s)x + sN(x)$ for all $s \in S$ and $x \in \mathcal{M}$. N is said to satisfy *Griffiths Transversality* if $N(\mathrm{Fil}^{i+1}\mathcal{M}) \subset \mathrm{Fil}^i\mathcal{M}$ for all $i \in \mathbb{Z}$.

Using operator N and $\mathrm{Fil}^i\mathcal{M}$, one can define another two filtration on $\mathcal{M}$. Set $\mathrm{F}^i\mathcal{M} = \mathrm{Fil}^i\mathcal{M}$ for $i \geq r$ and $\mathrm{F}^i\mathcal{M} = \mathcal{M}$ for $i \leq 0$; For $1 < i < r$, we inductively (start from $i = r-1$) define $\mathrm{F}^i\mathcal{M}$ to be the S-submodule generated by $N(\mathrm{F}^{i+1}\mathcal{M})$, $\mathrm{F}^{i+1}\mathcal{M}$ and $\mathrm{Fil}^i S\mathcal{M}$. Let M_K be $\mathcal{M}/\mathrm{Fil}^1 S\mathcal{M}$. Then M_K is a finite $\mathcal{O}_K$-module. Let f_π be the natural projection $f_\pi : \mathcal{M} \twoheadrightarrow M_K$. Then $\mathrm{Fil}^i M_K := f_\pi(\mathrm{Fil}^i\mathcal{M})$ defines a natural filtration on M_K. Define $\tilde{F}^i\mathcal{M} = \mathcal{M}$ for $i \leq 0$ and $\tilde{F}^i$ inductively by the following formula:

$$\tilde{F}^{i+1}\mathcal{M} := \{x \in \mathcal{M} | f_\pi(x) \in \mathrm{Fil}^{i+1} M_K, N(x) \in \tilde{F}^i\mathcal{M}\}.$$

Since N satisfies Griffiths Transversality, we have $\mathrm{F}^i\mathcal{M} \subset \mathrm{Fil}^i\mathcal{M} \subset \tilde{F}^i\mathcal{M}$ for all $i \in \mathbb{Z}$. It is easy to check by induction that $\mathrm{F}^i\mathcal{M}$ and $\tilde{F}^i\mathcal{M}$ satisfy the filtration condition. The following proposition shows that these three different filtration are not very different.

Proposition 2.2.1. *There exist constants c_1 and c_2 only depending on $E(u)$ and r, such that $p^{c_1}\mathrm{Fil}^i\mathcal{M} \subset \mathrm{F}^i\mathcal{M}$ and $p^{c_2}\tilde{F}^i\mathcal{M} \subset \mathrm{Fil}^i\mathcal{M}$ for all $i \in \mathbb{Z}$.*

Proof. It is easy to see that $N_S(E(u))$ and $E(u)$ are relative prime in $K_0[u]$. So there exists a constant γ such that $p^\gamma\mathfrak{S}$ is contained in the ideal generated by $N_S(E(u))$ and $E(u)$. Set $\alpha_i = \beta^{(i)}_{r-i}$ and $\beta^{(i)}_l = v_p(r-i-l+1) + \gamma + \beta^{(i)}_{l-1} + \alpha_{i-1}$ with $0 \leq l \leq r-i$, $\beta^{(i)}_0 = 0$ for all i and $\alpha_0 = 0$.

We will prove by induction on i that $p^{\alpha_i}\mathrm{Fil}^{r-i}\mathcal{M} \subset \mathrm{F}^{r-i}\mathcal{M}$ for $0 \leq i \leq r$. By definition, the statement is trivial when $i = 0$. Now assume that $i = j-1$ the statement is true, that is, $p^{\alpha_{j-1}}\mathrm{Fil}^{r-j+1}\mathcal{M} \subset \mathrm{F}^{r-j+1}\mathcal{M}$. Set $s = r-j$. For any $x \in \mathrm{Fil}^s\mathcal{M}$, by Griffiths Transversality, $E(u)^{s-l}N^{s-l}(x) \in \mathrm{Fil}^s\mathcal{M}$ for $0 \leq l \leq r-j$. We show by induction on l that $p^{\beta_l^{(j)}}E(u)^{s-l}N^{s-l}(x) \in \mathrm{F}^s\mathcal{M}$. If $l = 0$ then $E(u)^sN^s(x) \in E(u)^s\mathcal{M} \subset \mathrm{F}^s\mathcal{M}$ by definition. Now assume that the statement is valid for $l-1$, that is, $p^{\beta_{l-1}^{(j)}}E(u)^{s-l+1}N^{s-l+1}(x) \in \mathrm{F}^s\mathcal{M}$. Note that $E(u)E(u)^{s-l}N^{s-l}(x) \in \mathrm{Fil}^{s+1}\mathcal{M}$. Hence $p^{\alpha_{j-1}}N(E(u)^{s-l+1}N^{s-l}(x))$ is in $\mathrm{F}^s\mathcal{M}$ by the induction on i. On the other hand,

$$N(E(u)^{s-l+1}N^{s-l}(x)) = (s-l+1)N(E(u))E(u)^{s-l}N^{s-l}(x) + E(u)^{s-l+1}N^{s-l+1}(x).$$

By induction on l, we conclude that $p^{\lambda}(s-l+1)N(E(u))E(u)^{s-l}N^{s-l}(x) \in \mathrm{F}^s\mathcal{M}$ where $\lambda = \alpha_{j-1} + \beta_{l-1}^{(j)}$. Note that $p^{\alpha_{j-1}}E(u)E(u)^{s-l}N^{s-l}(x) \in \mathrm{F}^s\mathcal{M}$. So we have $p^{\lambda+\gamma+v_p(s-l+1)}E(u)^{s-l}N^{s-l}(x) \in \mathrm{F}^s\mathcal{M}$. Thus we prove that $p^{\beta_l^{(j)}}E(u)^{s-l}N^{s-l}(x) \in \mathrm{F}^s\mathcal{M}$. So $p^{\beta_{r-j}^{(j)}}x \in \mathrm{F}^s\mathcal{M}$, and then $p^{\alpha_j}\mathrm{Fil}^j\mathcal{M} \subset \mathrm{F}^j\mathcal{M}$. Now it suffices to set $c_1 := \sum_{j=0}^{r}\alpha_j$.

Now let us prove by induction on i that there exists constant μ_i depending on $E(u)$ and r such that $p^{\mu_i}\tilde{F}^i\mathcal{M} \subset \mathrm{Fil}^i\mathcal{M}$. By definition, $\tilde{F}^0\mathcal{M} = \mathrm{Fil}^0\mathcal{M}$. So μ_0 can be assigned to 0. Now assume that statement is valid for $i = j-1$. That is, there exists μ_{j-1} such that $p^{\mu_{j-1}}\tilde{F}^{j-1}\mathcal{M} \subset \mathrm{Fil}^{j-1}\mathcal{M}$. Now let us consider the case $i = j$. For any $x \in \tilde{F}^j\mathcal{M}$, we have $f_\pi(x) \in \mathrm{Fil}^jM_K$. Hence there exists a $y' \in \mathrm{Fil}^j\mathcal{M}$, $g \in \mathrm{Fil}^1S$ and $z' \in \mathcal{M}$ such that $x = y' + gz'$. Note that for any $g \in S$ there exists a constant λ only depending on r such that $p^{\lambda}g = g_0 + g_1$ with $g_0 \in W(k)[u]$ and $g_1 \in \mathrm{Fil}^rS$. So there exists a $y \in \mathrm{Fil}^r\mathcal{M}, z \in \mathcal{M}$ such that $p^{\lambda}x = y + E(u)z$. Now we claim that there exist constants ν_l such that $p^{\nu_l}f_\pi(N^l(z)) \in \mathrm{Fil}^{j-1-l}M_K$ for $0 \leq l \leq j-1$. Accept the claim for a while and set $\nu = \mathrm{Max}_l\{\nu_l\}$. We see that $f_\pi(N^l(p^{\nu}z)) \in \mathrm{Fil}^{j-1-l}M_K$ for $0 \leq l \leq j-1$. By definition of $\tilde{F}^i\mathcal{M}$, we easily see (by induction on l) that $N^l(p^{\nu}z) \in \tilde{F}^{j-1-l}\mathcal{M}$. In particular, $p^{\nu}z \in \tilde{F}^{j-1}\mathcal{M}$. By induction, we have $p^{\nu+\mu_{j-1}}z \in \mathrm{Fil}^{j-1}\mathcal{M}$. So set $\mu_j = \mu_{j-1} + \nu + \lambda$, we have $p^{\mu_j}x = p^{\nu+\mu_{j-1}}y + E(u)p^{\nu+\mu_{j-1}}z \in \mathrm{Fil}^j\mathcal{M}$.

Now it suffices to show that there exist ν_l such that $p^{\nu_l}f_\pi(N^l(z)) \in \mathrm{Fil}^{j-1-l}M_K$ for $0 \leq l \leq j-1$. We prove by induction on l. Let $l = 0$. Note that $N(p^{\lambda}x) = N(y) + E(u)N(z) + N(E(u))z$ is in $\tilde{F}^{j-1}\mathcal{M}$. By Griffiths Transversality, we see that $N(y)$ is in $\mathrm{Fil}^{j-1}\mathcal{M}$. Note that $f_\pi(E(u)N(z)) = 0$. So we have $f_\pi(N(E(u))z) \in \mathrm{Fil}^{j-1}M_K$. Let δ be the least integer not less than $v_p(N(E(u))(\pi))$. We see that $p^{\delta}f_\pi(z) \in \mathrm{Fil}^{j-1}M_K$. For a general l, we have

$$N^{l+1}(p^{\lambda}x) = N^{l+1}(y) + \sum_{m=0}^{l+1}\binom{l+1}{m}N^{l+1-m}(E(u))N^m(z).$$

By definition of $\tilde{F}^i\mathcal{M}$ and Griffiths Transversality, we have $N^{l+1}(p^{\lambda}x) \in \tilde{F}^{j-1-l}\mathcal{M}$ and $N^{l+1}(y) \in \mathrm{Fil}^{j-1-l}\mathcal{M}$. So applying f_π to the above equation, noting that $f_\pi(E(u)) = 0$, we have

$$\sum_{m=0}^{l}\binom{l+1}{m}f_\pi(N^{l+1-m}(E(u)))f_\pi(N^m(z)) \in \mathrm{Fil}^{j-1-l}M_K.$$

By induction, for $m = 0, \dots, l-1$, we have $p^{\nu_m} f_\pi(N^m(z)) \in \mathrm{Fil}^{j-1-m} M_K \subset \mathrm{Fil}^{j-1-l} M_K$. Let $\tilde{\nu} = \max\{\nu_m | m = 0, \dots, l-1.\}$. We conclude that

$$p^{\tilde{\nu}}(l+1) f_\pi(N(E(u))) f_\pi(N^l(z)) \in \mathrm{Fil}^{j-1-l} M_K.$$

By setting $\nu_l = \tilde{\nu} + v_p(l+1) + \delta$ we have $p^{\nu_l} f_\pi(N^l(z)) \in \mathrm{Fil}^{j-1-l} M_K$. This completes the induction and proves the claim. □

2.3. **Filtration from Kisin modules.** Now let us study how the Frobenius on Breuil modules interacts with filtration. In particular, we discuss filtration built from Frobenius of Kisin modules. For this, we define the following:

A filtered φ-module $\mathcal{M}$ over S is an S-module $\mathcal{M}$ with

(1) a φ_S-semi-linear morphism $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$.
(2) a decreasing filtration $\mathrm{Fil}^i \mathcal{M} \subset \mathcal{M}$ satisfying the filtration conditions.

A filtered φ-module $\mathcal{D}$ over S_{K_0} or a *Breuil module $\mathcal{D}$* is a filtered φ-module over S such that

(1) $\mathcal{D}$ is finite S_{K_0}-free and the determinant of $\varphi_{\mathcal{D}}$ is invertible in S_{K_0}
(2) There exists a monodromy operator $N : \mathcal{D} \to \mathcal{D}$ satisfying Griffiths Transversality and $N_{\mathcal{D}} \varphi_{\mathcal{D}} = p \varphi_{\mathcal{D}} N_{\mathcal{D}}$.

Let D be a filtered (φ, N)-module over K_0. Following [Bre97a], we can associate a Breuil module as following: $\mathcal{D} = S \otimes_{K_0} D$, $\varphi_{\mathcal{D}} := \varphi_S \otimes \varphi_D$, $N_{\mathcal{D}} = N_S \otimes \mathrm{Id} + \mathrm{Id} \otimes N_D$, $\mathrm{Fil}^0 \mathcal{D} := \mathcal{D}$ and by induction

$$\mathrm{Fil}^{i+1} \mathcal{D} := \{x \in \mathcal{D} | N(x) \in \mathrm{Fil}^i \mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^{i+1} D_K\},$$

where $f_\pi : \mathcal{D} \to D_K$ is the natural projection defined by $\mathcal{D} \to \mathcal{D}/\mathrm{Fil}^1 S \mathcal{D} \simeq D_K$. Conversely, given a Breuil module $\mathcal{D}$, one can recover D via $D := \mathcal{D}/I_+ S \mathcal{D}$; $\varphi_D := \varphi_{\mathcal{D}} \mod I_+ S \mathcal{D}$; $N_D := N_{\mathcal{D}} \mod I_+ S \mathcal{D}$ and $\mathrm{Fil}^i D_K := f_\pi(\mathrm{Fil}^i \mathcal{D})$. The main theorem in [Bre97a] showed that the functor $D \mapsto \mathcal{D}(D)$ is an equivalence of categories between the category of filtered (φ, N)-modules over K_0 with $\mathrm{Fil}^0 D_K = D_K$ and the category of Breuil modules.

Now let us recall Kisin module and its basic properties as in [Liu07] and [Liu10]. Recall $\mathfrak{S} := W(k)[\![u]\!]$ with a Frobenius endomorphism φ via $u \mapsto u^p$ and the natural Frobenius on $W(k)$, and the category of φ-modules over $\mathfrak{S}$. Denote by $'\mathrm{Mod}^{\varphi, r}_{/\mathfrak{S}}$ the category of *φ-modules of height r*, in the sense that $\mathfrak{M}$ is $\mathfrak{S}$-finite type and the cokernel of φ^* is killed by $E(u)^r$, where φ^* is the $\mathfrak{S}$-linear map $1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$. By definition, a *finite free Kisin module* (of height r) is a φ-module (of height r) $\mathfrak{M}$ such that $\mathfrak{M}$ is finite $\mathfrak{S}$-free. A *torsion Kisin module* (of height r) is a φ-module $\mathfrak{M}$ such that $\mathfrak{M}$ is killed by some p-power and there exists an injective morphism $\mathfrak{L} \subset \mathfrak{L}'$ of two finite free Kisin modules of height r satisfying $\mathfrak{M} \simeq \mathfrak{L}'/\mathfrak{L}$. When we mention Kisin module (of height r) in the remaining of the paper, we mean either finite free Kisin module of height r or torsion Kisin module of height r.

Let $K_\infty := \bigcup\limits_{n=0}^{\infty} K(\sqrt[p^n]{\pi})$ and $G_\infty = \mathrm{Gal}(\overline{K}/K_\infty)$. There exists a functor $T_{\mathfrak{S}}$ from the category of Kisin modules to the category of $\mathbb{Z}_p[G_\infty]$-modules: If $\mathfrak{M}$ is finite free then $T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, W(R))$ and if $\mathfrak{M}$ is p-power torsion then $T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\varphi, \mathfrak{S}}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))$, where $W(R)$ is an $\mathfrak{S}$-algebra with a natural Frobenius and a natural G-action. Though $T_{\mathfrak{S}}$ has many nice properties, we do not need them in this paper. The readers are refereed to [Liu07] for the construction of $W(R)$ and more discussion of $T_{\mathfrak{S}}$.

Let $(\mathfrak{M}, \varphi)$ be a Kisin module. Following §5.3 in [Liu07], we can define a functor $\mathcal{M}_S$ from the category of Kisin modules to the category of filtered φ-modules over S as the following: Define $\mathcal{M}_S(\mathfrak{M}) = S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ and $\varphi_{\mathcal{M}_S(\mathfrak{M})} := \varphi_S \otimes \varphi_{\mathfrak{M}}$; Note that $1 \otimes \varphi : \mathcal{M}_S(\mathfrak{M}) \to S \otimes_{\mathfrak{S}} \mathfrak{M}$ is an S-linear map. Set

$$\mathcal{F}^i \mathcal{M}_S(\mathfrak{M}) := \{m \in \mathcal{M}_S(\mathfrak{M}) | (1 \otimes \varphi)(m) \in \mathrm{Fil}^i S \otimes_S \mathfrak{M}\}. \tag{2.3.1}$$

Lemma 2.3.1. *Assume that $\mathfrak{M}$ is finite $\mathfrak{S}$-free and write $\mathcal{M} = \mathcal{M}_S(\mathfrak{M})$. Then $\mathcal{F}^i \mathcal{M}$ satisfies the filtration condition defined in the previous subsection.*

Proof. All other requirements are easily to verified by the definition except that $\mathcal{F}^{r+1} \mathcal{M} \subset \mathrm{Fil}^1 S \mathcal{M}$. Let $\{e_1, \dots, e_d\}$ be an $\mathfrak{S}$-basis of $\mathfrak{M}$. Assume that $x = \sum_i a_i \otimes e_i$ is in $\mathcal{F}^{r+1}\mathcal{M}$ with $a_i \in S$. We have to show that $a_i \in \mathrm{Fil}^1 S$. Let A be a matrix in $\mathrm{M}_d(\mathfrak{S})$ such that $(\varphi(e_1), \dots, \varphi(e_d)) = (e_1, \dots, e_d)A$. Since $(1 \otimes \varphi)(x) = \sum_i a_i \varphi(e_i)$ is in $\mathrm{Fil}^{r+1} S \otimes_{\mathfrak{S}} \mathfrak{M}$, we conclude that $A\alpha = \beta$ where α, β are $n \times 1$ matrices, coefficients of α are a_i, and coefficients of β are in $\mathrm{Fil}^{r+1} S$. The fact that $\mathfrak{M}$ has $E(u)$-height r means that there exists a matrix $B \in \mathrm{M}_d(\mathfrak{S})$ such that $AB = BA = E(u)^r I_d$. Hence $B\beta = BA\alpha = E(u)^r \alpha$ still has coefficients in $\mathrm{Fil}^{r+1} S$, and then α has all its coefficients in $\mathrm{Fil}^1 S$. □

2.4. **Lattices in $D_{\mathrm{dR}}(V)$.** Let V be a de Rham representation of G. We denote $D_{\mathrm{dR}}(V) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^\vee)^G$. Now let us summarize results from [Liu10] and [Liu11] to manipulate lattices in semi-stable representations. Let V be a semi-stable representation in $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}, r}$, $\Lambda \subset V$ a G-stable $\mathbb{Z}_p$-lattice. The main result of [Liu10] is that there exists an anti-equivalence $\hat{T}$ of categories between $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{st}, r}$ and the category of $(\varphi, \hat{G})$-modules of height r. Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ be the $(\varphi, \hat{G})$-module such that $\hat{T}(\hat{\mathfrak{M}}) \simeq \Lambda$ with $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ the ambient finite free Kisin module of height r. In particular, this means that $\mathfrak{M}$ is the unique finite free Kisin module such that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq \Lambda|_{G_\infty}$ (see Theorem 2.3.1 in [Liu10]). Let $D = D_{\mathrm{st}}(V)$ the filtered (φ, N)-module attached to V. Set $\mathcal{D} := \mathcal{D}(D)$ the Breuil module associated to D and $\mathcal{M} := \mathcal{M}_S(\mathfrak{M})$ the filtered φ-modules over S. There exists a natural isomorphism of φ-modules over S between $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{M}$ and $\mathcal{D}$ (see §3 in [Liu08] for full details). In particular, $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{M}$ has the structure of monodromy N. By Proposition 2.4.1 in [Liu11], we have $N(\mathcal{M}) \subset \mathcal{M}$ if $p > 2$ [4]. Let $\mathrm{M}_S(\varphi, N, \mathrm{Fil})$ the full subcategory of filtered φ-modules $\mathcal{M}$ over S such that

- There exists a finite free Kisin module $\mathfrak{M}$ such that $\mathcal{M} \simeq \mathcal{M}_S(\mathfrak{M})$.
- $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{M}$ has a structure of Breuil module and $N(\mathcal{M}) \subset \mathcal{M}$.

Hence we obtain a contravariant functor T_{st}^{-1} from $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{st}, r}$ to $\mathrm{M}_S(\varphi, N, \mathrm{Fil})$. If $r < p-1$ then T_{st}^{-1} is an anti-equivalence by the main result of [Liu08]. But T_{st}^{-1} in general is not a full functor if $r \geq p-1$. We easily check that $\mathfrak{M}/u\mathfrak{M} \simeq \mathcal{M}/I_+ S\mathcal{M}$ as φ-modules and $f_\pi(\mathcal{M}) = \mathcal{M}/\mathrm{Fil}^1 S\mathcal{M}$ is an $\mathcal{O}_K$-lattice in $\mathcal{D}/\mathrm{Fil}^1 S\mathcal{D}$.

Now let us recall a little more details on the construction of $M_{\mathrm{st}}(\Lambda)$ from §2.2 and §2.3 from [Liu11], let $\tilde{D} := \mathcal{D}/I_+ SD$, which is a φ-module over K_0. Then $M := \mathfrak{M}/u\mathfrak{M} \simeq \mathcal{M}/I_+ S\mathcal{M}$ is a φ-stable $W(k)$-lattice in $\tilde{D}$. Note that $\tilde{D}$ is canonically

[4] Here is the only place that we need p to be an odd prime.

isomorphic to $D = D_{\mathrm{st}}(V)$ via the unique φ-compatible section $s : \tilde{D} \hookrightarrow \mathcal{D}$ (see Proposition 6.2.1.1 in [Bre97b]). Then $M_{\mathrm{st}}(\Lambda)$ is just the image $s(M)$.

Now identify $\mathcal{D}/\mathrm{Fil}^1 S\mathcal{D}$ with $D_K = D_{\mathrm{dR}}(V)$ via $\mathcal{D} \simeq S \otimes_{W(k)} D$. We obtain two $\mathcal{O}_K$-lattices: $M_{\mathrm{st}}(\Lambda)_K := \mathcal{O}_K \otimes_{W(k)} s(M)$ and $f_\pi(\mathcal{M})$.

Proposition 2.4.1. *There exists a constant c_3 only depending on e and r such that $p^{c_3}(M_{\mathrm{st}}(\Lambda)_K) \subset f_\pi(\mathcal{M})$ and $p^{c_3}(f_\pi(\mathcal{M})) \subset M_{\mathrm{st}}(\Lambda)_K$.*

Proof. Note that $\mathcal{M} = S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ with a Kisin module $\mathfrak{M}$ of height r. If $\{\tilde{e}_1, \dots, \tilde{e}_d\}$ is an $\mathfrak{S}$-basis of $\mathfrak{M}$. Then $\{\hat{e}_i := (1 \otimes \varphi)(\tilde{e}_i)\}$ forms an S-basis of $\mathcal{M}$. Let e_i be the image of $\hat{e}_i$ of the natural map $\mathcal{M} \to \mathcal{M}/I_+ S\mathcal{M} = M$. Since $\mathcal{M}/I_+S\mathcal{M}$ has a unique φ-equivariant section $s : M \to \mathcal{D}$, we just denote e_i for $s(e_i)$. Note that $e_1, \dots, e_d$ forms a basis of $s(M)$. Let $A \in \mathrm{M}_d(\mathfrak{S})$ be the matrix such that $\varphi(\hat{e}_1, \dots, \hat{e}_d) = (\hat{e}_1, \dots, \hat{e}_d)A$ and $A_0 := A \mod u$ the matrix in $\mathrm{M}_d(W(k))$. Then we have $\varphi(e_1, \dots, e_d) = (e_1, \dots, e_d)A_0$. Let $X \in \mathrm{M}_d(S_{K_0})$ be the matrix such that $(\hat{e}_1, \dots, \hat{e}_d) = (e_1, \dots, e_d)X$. It suffices to show that there exists a constant c_3 such that $p^{c_3}X$ and $p^{c_3}X^{-1}$ are in $\mathrm{M}_d(S)$. To proceed the proof, note that we have the relation $A_0\varphi(X) = XA$. Set

$$X_n := A_0\varphi(A_0)\cdots\varphi^n(A_0)\varphi^n(A^{-1})\cdots\varphi(A^{-1})A^{-1}.$$

Following the same idea of the proof of Proposition 6.2.1.1 in [Bre97b], we show that X_n converges to X and there exists a constant c_3 such that $p^{c_3}X_n \in \mathrm{M}_d(S)$ and hence $p^{c_3}X \in \mathrm{M}_d(S)$. To prove this, we first claim that $p^rA^{-1} \in \mathrm{M}_d(S)$ and $A_0A^{-1} = I_d + \frac{u^p}{p^r}Y$ with $Y \in \mathrm{M}_d(S)$. We accept this claim and postpone the proof in the end. Set $c_3 = \max_{i \geq 0}(ri - v_p(q(p^i)!))$ where $q(p^i)$ satisfies the relation $p^i = eq(p^i) + r(p^i)$ with $0 \leq r(p^i) < e$. Now $X_n = X_0 + \sum\limits_{i=0}^{n-1}(X_{i+1} - X_i) = X_0 + \sum\limits_{i=0}^{n-1} \frac{u^{p^{i+2}}}{p^r} Z_i$ where

$$Z_i = A_0\varphi(A_0)\cdots\varphi^i(A_0)\varphi^{i+1}(Y)\varphi^i(A^{-1})\cdots\varphi(A^{-1})A^{-1}.$$

Since $p^rA^{-1} \in \mathrm{M}_d(S)$, we see that $\frac{u^{p^{i+2}}}{p^r}Z_i = \frac{u^{p^{i+2}}}{p^{r(i+2)}}p^{r(i+1)}Z_i$ is in $\frac{u^{p^{i+2}}}{p^{r(i+2)}}\mathrm{M}_d(S)$. As any $a \in S$ can be (uniquely) written as $\sum\limits_{i=0}^{\infty} a_i \frac{u^i}{q(i)!}$ with $a_i \in W(k)$, we conclude that $p^{c_3}\frac{u^{p^{i+2}}}{p^{r(i+2)}}$ is in S. Hence $p^{c_3}X_n$ and then $p^{c_3}X$ are in $\mathrm{M}_d(S)$.

Now let us prove the claim. Since $\mathcal{M} = S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ with a Kisin module $\mathfrak{M}$ of height r, we have $A = \varphi(\tilde{A})$ with $\tilde{A} \in \mathrm{M}_d(\mathfrak{S})$ and there exists a matrix $\tilde{B} \in \mathrm{M}_d(\mathfrak{S})$ such that $\tilde{A}\tilde{B} = \tilde{B}\tilde{A} = E(u)^rI_d$. Hence $A^{-1} = \varphi(\tilde{A}^{-1}) = (\varphi(E(u)^r)^{-1}\varphi(\tilde{B})$. Note that $\varphi(E(u))/p$ is a unit in S. Hence p^rA^{-1} is in $\mathrm{M}_d(S)$. Now $A_0A^{-1} = A_0\varphi(\tilde{A}^{-1}\tilde{B}^{-1})\varphi(\tilde{B}) = (\varphi(E(u)^r))^{-1}A_0\varphi(\tilde{B})$. Let pa_0 be the constant term of $E(u)$. It is easy to see that $A_0\varphi(\tilde{B}) = (a_0p)^rI_d + u^pY'$ with $Y' \in \mathrm{M}_d(\mathfrak{S})$. Now we have $A_0A^{-1} = I_d + (b^r - 1)I_d + \frac{u^p}{p^r}(a_0b)^rY'$ with $b = (\varphi(E(u))/c_0p)^{-1} \in S^\times$. We easily compute that $b = 1 + b'$ with $b' \in \frac{u^{ep}}{p}S$. Hence $A_0A^{-1} = I_d + \frac{u^p}{p^r}Y$ with $Y \in \mathrm{M}_d(S)$.

Finally, it remains to show that $p^{c_3}X^{-1}$ is in $\mathrm{M}_d(S)$. In fact, we can use the same strategy to X_n^{-1}. In this situation, we need to show that $p^rA_0^{-1} \in \mathrm{M}_d(W(k))$ and $AA_0^{-1} = I_d + \frac{u^p}{p^r}Y$ with $Y \in \mathrm{M}_d(\mathfrak{S})$ and this is easy to show by a similar argument as the above. $\square$

The following example shows that $c_3 \neq 0$ in general.

Example 2.4.2. Let $\mathfrak{M}$ be a finite free rank-2 Kisin module of height 1 given by

$$\varphi(e_1, e_2) = (e_1, e_2)\begin{pmatrix} 1 & u \\ 0 & E(u) \end{pmatrix}$$

where e_1, e_d forms an $\mathfrak{S}$-basis of $\mathfrak{M}$. By Theorem (0.4) in [Kis06], $\mathfrak{M}$ corresponds to a $\mathbb{Z}_p$-lattice of crystalline representation with Hodge-Tate weight in $\{0, 1\}$. Using notations in the above proof we have $A = \begin{pmatrix} 1 & u^p \\ 0 & \varphi(E(u)) \end{pmatrix}$ and $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Write $X = \begin{pmatrix} 1 & x \\ 0 & \alpha \end{pmatrix}$. Then the relation $A_0\varphi(X) = XA$ yields two equations: $p\varphi(\alpha) = \alpha\varphi(E(u))$ and $\varphi(x) = u^p + x\varphi(E(u))$. Since $\varphi(E(u)) = p\mu$ with μ a unit in S, we easily solve that α is a unit of S and $x = 1 + \frac{u^p}{p} +$ higher degree term. If $e > 1$, we see that u^p/p is not in S. And if $e > p$ then $x \mod \mathrm{Fil}^1 S$ is not in $\mathcal{O}_K$. So $M_{\mathrm{st}}(\Lambda)_K$ and $f_\pi(\mathcal{M})$ are different $\mathcal{O}_K$-lattices.

From the construction of $\mathcal{F}^i\mathcal{M}_S(\mathfrak{M})$, we easily see that $\mathcal{F}^i\mathcal{M}_S(\mathfrak{M}) = \mathcal{M} \cap \mathrm{Fil}^i\mathcal{D}$. So we may just denote $\mathcal{F}^i\mathcal{M}$ by $\mathrm{Fil}^i\mathcal{M}$. Consider the natural projection $f_\pi : \mathcal{D} \twoheadrightarrow D_K$. Write $\tilde{M}_K = f_\pi(\mathcal{M})$ and define $\mathrm{Fil}^i\tilde{M}_K := \tilde{M}_K \cap \mathrm{Fil}^i D_K$. Obviously, $f_\pi(\mathrm{Fil}^i\mathcal{M}) \subset \mathrm{Fil}^i\tilde{M}_K$.

Lemma 2.4.3. *There exists a constant c_4 only depending on $E(u)$ and r such that $p^{c_4}\mathrm{Fil}^i\tilde{M}_K \subset f_\pi(\mathrm{Fil}^i\mathcal{M})$.*

Proof. Here we modify the idea used in the proof of Proposition 6.2.2.3 in [Bre97a]. We prove by induction on i that there exists a constant μ_i depending on $E(u)$ and i such that $p^{\mu_i}\mathrm{Fil}^i\tilde{M}_K \subset f_\pi(\mathrm{Fil}^i\mathcal{M})$. If $i = 0$ the case is trivial. Now assume the statement is valid for any $i \leq j$. Without loss of generality, we may assume that $\mu_{i-1} \leq \mu_i$ for any $1 \leq i \leq j$. Now consider the case $i = j + 1$. Let $x \in \mathrm{Fil}^{j+1}\tilde{M}_K$ then $x \in \mathrm{Fil}^j\tilde{M}_K$. By induction, $p^{\mu_j}x \in f_\pi(\mathrm{Fil}^j\mathcal{M})$. That is, there exists a $\hat{x} \in \mathrm{Fil}^j\mathcal{M}$ such that $f_\pi(\hat{x}) = p^{\mu_j}x$. Write $N(E(u)) = R(u)$. Note that $R(\pi) \neq 0$. So there exists $Q(u) \in K_0[u]$ such that $Q(\pi)R(\pi) = -1$. Let $H(u) = Q(u)E(u)$. Note that $1 + N(H(u)) \in \mathrm{Fil}^1 S_K$. Now set

$$\hat{y} := \hat{x} + H(u)N(\hat{x}) + \frac{1}{2}H^2(u)N^2(\hat{x}) + \cdots + \frac{1}{j!}H^j(u)N^j(\hat{x}).$$

It is obvious that $f_\pi(\hat{y}) = f_\pi(\hat{x})$. Write $m(u) = 1 + N(H(u))$, we have

$$N(\hat{y}) = m(u)N(\hat{x}) + \sum_{i=1}^{j-1}\frac{1}{i!}m(u)(H(u))^i N^{i+1}(\hat{x}) + \frac{1}{j!}(H(u)^j)N^{j+1}(\hat{x}).$$

Since $N^i(\hat{x}) \in \mathrm{Fil}^{j-i}\mathcal{M}$ for $1 \leq i \leq j$, $m(u) \in \mathrm{Fil}^1 S_{K_0}$ and $H(u) \in \mathrm{Fil}^1 S_{K_0}$, we see that $N(\hat{y}) \in \mathrm{Fil}^j\mathcal{D}$. Hence $\hat{y} \in \mathrm{Fil}^{j+1}\mathcal{D}$. Let λ be the minimal constant such that $p^\lambda Q(u) \in W(k)[u]$. Apparently λ only depends on $E(u)$. Now $p^{\lambda j}(H(u))^j \in E(u)^j W(k)[u]$ and then $p^{\lambda j}\hat{y} \in \mathcal{M}$. Hence $p^{\lambda j}\hat{y} \in \mathrm{Fil}^{j+1}\mathcal{M}$. So set $\mu_{j+1} = \mu_j + \lambda j$. We see that $f_\pi(p^{\lambda j}\hat{y}) = p^{\lambda j}f_\pi(\hat{x}) = p^{\mu_{j+1}}x$. □

Remark 2.4.4. The constant c_3 and c_4 are not optimal. In fact, assume that representations are crystalline, $K_0 = K$ and $0 \leq r \leq p - 2$. One can choose that

$c_3 = c_4 = 0$ by using Fontaine-Laffile theory in [FL82]. But in general, we do not expect c_3 or c_4 is zero.

3. Filtration Attached to Torsion Semi-stable Representations

3.1. **Construction of filtration to torsion representations.** Now let us first discuss more details on filtration associated to torsion semi-stable representations. Let $T \in \mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor}}$ be a torsion semi-stable representation and $j : \Lambda \hookrightarrow \Lambda^\star$ is a lift of T. That is, $j : \Lambda \subset \Lambda^\star$ are G-stable $\mathbb{Z}_p$-lattices inside a semi-stable representation with Hodge-Tate weights in $\{0, \dots, r\}$ and we have the exact sequence of $\mathbb{Z}_p[G]$-modules $0 \to \Lambda \overset{j}{\to} \Lambda^\star \overset{q}{\to} T \to 0$. Recall that $\hat{T}$ is an anti-equivalence between the category of $(\varphi, \hat{G})$-modules of height r and $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p}$. Let $\mathfrak{L}$ and $\mathfrak{L}^\star$ be the ambient Kisin modules of $(\varphi, \hat{G})$-modules correspond to Λ and $\Lambda^\star$ respectively, we obtain the injective morphism of Kisin modules $\mathfrak{j} : \mathfrak{L}^\star \hookrightarrow \mathfrak{L}$. Write $\mathfrak{M} := \mathfrak{L}/\mathfrak{j}(\mathfrak{L}^\star)$ which is a torsion Kisin module of height r. By Proposition 3.2.3 in [Liu11], we have $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T|_{G_\infty}$.

Now consider the exact sequence of Kisin modules to correspond the above exact sequence of Galois representations:

$$0 \longrightarrow \mathfrak{L}^\star \xrightarrow{\ \mathfrak{j}\ } \mathfrak{L} \xrightarrow{\ \mathfrak{q}\ } \mathfrak{M} \longrightarrow 0. \tag{3.1.1}$$

Now modulo u, we have an exact sequence

$$0 \longrightarrow L^\star \xrightarrow{\ \bar{\mathfrak{j}}\ } L \xrightarrow{\ \bar{\mathfrak{q}}\ } M \longrightarrow 0\ , \tag{3.1.2}$$

By the construction of M_{st}, the exact sequence (3.1.2) is canonically isomorphic to the exact sequence

$$0 \longrightarrow M_{\mathrm{st}}(\Lambda^\star) \xrightarrow{M_{\mathrm{st}}(j)} M_{\mathrm{st}}(\Lambda) \longrightarrow M_{\mathrm{st},j}(T) \longrightarrow 0\ .$$

By tensoring $\mathcal{O}_K$ to the above exact sequence, we obtain an exact sequence of $\mathcal{O}_K$-modules

$$0 \longrightarrow L^\star_K \xrightarrow{\ \bar{\mathfrak{j}}_K\ } L_K \xrightarrow{\ \bar{\mathfrak{q}}_K\ } M_K \longrightarrow 0\ , \tag{3.1.3}$$

Recall that $\mathrm{Fil}^i L_K = L_K \cap \mathrm{Fil}^i D_K$ where $D_K := D_{\mathrm{dR}}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda)$. For any $i \in \mathbb{Z}$, by the construction in §2.1, we have $\mathrm{Fil}^i M_K := \bar{\mathfrak{q}}_K(\mathrm{Fil}^i L_K)$ and the following *exact* sequence

$$0 \longrightarrow \mathrm{Fil}^i L^\star_K \xrightarrow{\ \bar{\mathfrak{j}}_K\ } \mathrm{Fil}^i L_K \xrightarrow{\ \bar{\mathfrak{q}}_K\ } \mathrm{Fil}^i M_K \longrightarrow 0. \tag{3.1.4}$$

Using Snake Lemma, the above exact sequence induces the following exact sequence:

Corollary 3.1.1. *The following sequence is exact*

$$0 \longrightarrow \mathrm{gr}^i L^\star_K \xrightarrow{\ \bar{\mathfrak{j}}_K\ } \mathrm{gr}^i L_K \xrightarrow{\ \bar{\mathfrak{q}}_K\ } \mathrm{gr}^i M_K \longrightarrow 0.$$

3.2. **The proof of Theorem 2.1.3.** Now we need to recall a part of Theorem 3.1.1 in [Liu11] and its proof to complete the proof of Theorem 2.1.3. Let $\mathrm{M}_{\mathrm{tor}}(\varphi, N)$ whose objects are finite length $W(k)$-modules with only φ and N-structures satisfying the properties required in the definition of filtered (φ, N)-modules.

Theorem 3.2.1. *There exists a constant $\mathfrak{c}$ only depending on e and r such that the following statement holds: for any morphism $f : T' \to T$ in $\mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor}}$ and any lift j', j of T', T respectively, there exists a morphism $g : M_{\mathrm{st},j}(T) \to M_{\mathrm{st},j'}(T')$ in $\mathrm{M}_{\mathrm{tor}}(\varphi, N)$ such that*

(1) *if there exists a morphism of lifts $\hat{f} : j' \to j$ which lifts f then $g = p^{\mathfrak{c}} M_{\mathrm{st},\hat{f}}(f)$.*
(2) *let $f' : T'' \to T'$ be a morphism in $\mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor}}$ with j'' the lift of f' and $g' : M_{st,j'}(T') \to M_{\mathrm{st},j''}(T'')$ the morphism in $\mathrm{M}_{\mathrm{tor}}(\varphi, N)$ attached to f', j' and j''. If there exists a morphism of lifts $\hat{h} : j'' \to j$ which lifts $f \circ f'$ then $g' \circ g = p^{2\mathfrak{c}} M_{\mathrm{st},\hat{h}}(f \circ f')$.*

The above theorem is a part of Theorem 3.1.1 in [Liu11]. To prove Theorem 2.1.3, it suffices to show that there exists a constant c_5 only depending on $E(u)$ and r such that $\tilde{g} := p^{c_5}(\mathcal{O}_K \otimes_{W(k)} g)$ and $\tilde{g}' := p^{c_5}(\mathcal{O}_K \otimes_{W(k)} g')$ preserve filtration defined in the previous subsection.

Now let us recall the construction of g. Let $\mathfrak{M}$ be the torsion Kisin module obtained by the exact sequence (3.1.1). Proposition 3.2.3 in [Liu11] explains that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T|_{G_\infty}$. Similarly, the lift $j' : \Lambda' \hookrightarrow \Lambda'^{\star}$ of T' induces a torsion Kisin module $\mathfrak{M}'$ such that $T_{\mathfrak{S}}(\mathfrak{M}') \simeq T'|_{G_\infty}$. By Theorem 2.4.2 in [Liu07], there exists a unique morphism $\mathfrak{f} : \mathfrak{M} \to \mathfrak{M}'$ of Kisin modules such that $T_{\mathfrak{S}}(\mathfrak{f}) = p^{\mathfrak{c}} f$. Then g is constructed as $g := \mathfrak{f} \mod u\mathfrak{S}$.

Define a map $\mathfrak{i}_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{L} \oplus \mathfrak{L}'$ via $\mathfrak{i}_{\mathfrak{L}}(x) = (x, 0)$ and define $\mathfrak{i}_{\mathfrak{L}'} : \mathfrak{L}' \to \mathfrak{L} \oplus \mathfrak{L}'$ via $\mathfrak{i}_{\mathfrak{L}'}(y) = (0, y)$. Set $\tilde{\mathfrak{q}} : \mathfrak{L} \oplus \mathfrak{L}' \to \mathfrak{M}'$ via $(\mathfrak{f} \circ \mathfrak{q})(x) + \mathfrak{q}'(y)$ for $(x, y) \in \mathfrak{L} \oplus \mathfrak{L}'$ and $\mathfrak{N} := \mathrm{Ker}\tilde{\mathfrak{q}}$. By Corollary 2.3.8 in [Liu07], $\mathfrak{N}$ is a finite free Kisin module of height r. Then we have the following commutative diagram of Kisin modules.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{L}'^{\star} & \longrightarrow & \mathfrak{L}' & \xrightarrow{\mathfrak{q}'} & \mathfrak{M}' & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathfrak{i}_{\mathfrak{L}'}} & & \Big\downarrow{\scriptstyle \wr} & & \\
0 & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{L} \oplus \mathfrak{L}' & \xrightarrow{\tilde{\mathfrak{q}}} & \mathfrak{M}' & \longrightarrow & 0 \\
 & & \Big\uparrow & & {\scriptstyle \mathfrak{i}_{\mathfrak{L}}}\Big\uparrow & & {\scriptstyle \mathfrak{f}}\Big\uparrow & & \\
0 & \longrightarrow & \mathfrak{L}^{\star} & \longrightarrow & \mathfrak{L} & \xrightarrow{\mathfrak{q}} & \mathfrak{M} & \longrightarrow & 0
\end{array}
\tag{3.2.1}
$$

It is easy to check that by functor $T_{\mathfrak{S}}$ or $\hat{T}$, the above commutative diagram corresponds the following commutative diagram of Galois representations

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda' & \longrightarrow & \Lambda'^{\star} & \xrightarrow{q'} & T' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \wr & & \\
0 & \longrightarrow & \Lambda \oplus \Lambda' & \xrightarrow{\tilde{j}} & N & \xrightarrow{\tilde{q}} & T' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow p^{\mathfrak{c}} f & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & \Lambda^{\star} & \xrightarrow{q} & T & \longrightarrow & 0
\end{array}
\tag{3.2.2}
$$

The above diagram can be constructed without using the previous diagram as the following: Taking dual of the first row and the last row of the above diagram, we obtain two exact sequences:

$$0 \to (\Lambda'^{\star})^{\vee} \to (\Lambda')^{\vee} \xrightarrow{q'^{\vee}} (T')^{\vee} \to 0 \text{ and } 0 \to (\Lambda^{\star})^{\vee} \to \Lambda^{\vee} \xrightarrow{q^{\vee}} T^{\vee} \to 0.$$

We can construct $\tilde{q}^{\vee} : \Lambda^{\vee} \oplus (\Lambda')^{\vee} \to (T')^{\vee}$ by $\tilde{q}^{\vee}((x, y)) = p^{\mathfrak{c}} f^{\vee} \circ q^{\vee}(x) + {q'}^{\vee}(y)$ and let $N^{\vee} := \mathrm{Ker}(\tilde{q}^{\vee})$. We embed $\Lambda^{\vee}$ and ${\Lambda'}^{\vee}$ to $\Lambda^{\vee} \oplus (\Lambda')^{\vee}$ to the first factor and the second factor respectively, In this way, we obtain a commutative diagram as Diagram (3.2.1)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda'^{\star\vee} & \longrightarrow & \Lambda'^{\vee} & \xrightarrow{q'^{\vee}} & (T')^{\vee} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \wr & & \\
0 & \longrightarrow & N^{\vee} & \longrightarrow & \Lambda^{\vee} \oplus \Lambda'^{\vee} & \xrightarrow{\tilde{q}^{\vee}} & (T')^{\vee} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow p^{\mathfrak{c}} f^{\vee} & & \\
0 & \longrightarrow & (\Lambda^{\star})^{\vee} & \longrightarrow & \Lambda^{\vee} & \xrightarrow{q^{\vee}} & T^{\vee} & \longrightarrow & 0
\end{array}
\tag{3.2.3}
$$

Then Diagram (3.2.2) is obtained by taking dual of the above diagram. In summary, we obtain another lift $\tilde{j}$ of T'. It is obviously that the map $g : M_{\mathrm{st},j}(T) \to M_{\mathrm{st},\tilde{j}}(T')$ preserves filtration. We also have a map $\alpha : M_{\mathrm{st},j'}(T') \to M_{\mathrm{st},\tilde{j}}(T')$ by modulo u of the upper block of Diagram (3.2.1), which is a Frobenius, monodromy compatible isomorphism of $W(k)$-modules. It is clear that $\alpha_K(\mathrm{Fil}^i M'_{K,j'}) \subset \mathrm{Fil}^i M'_{K,\tilde{j}}$ where $\alpha_K := \mathcal{O}_K \otimes_{W(k)} \alpha$, $\mathrm{Fil}^i M'_{K,j'}$ and $\mathrm{Fil}^i M'_{K,\tilde{j}}$ are filtration of $\mathcal{O}_K \otimes_{W(k)} M_{\mathrm{st},j'}(T')$ and $\mathcal{O}_K \otimes_{W(k)} M_{\mathrm{st},\tilde{j}}(T')$ via the construction in the last subsection. Now to prove Theorem 2.1.3, it suffices to prove that there exists a constant c_5 only depending on $E(u)$ and r such that $p^{c_5} \mathrm{Fil}^i M'_{K,\tilde{j}} \subset \alpha_K(\mathrm{Fil}^i M'_{K,j'})$.

To prove this statement, consider the following commutative diagram

$$
(3.2.4)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{L}'^{\star} & \longrightarrow & \mathfrak{L}' & \xrightarrow{\mathfrak{q}'} & \mathfrak{M}' & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{i}_{\mathfrak{L}'}} & & \big\downarrow{\scriptstyle \wr} & & \\
0 & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{L}\oplus\mathfrak{L}' & \xrightarrow{\tilde{\mathfrak{q}}} & \mathfrak{M}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{\mathfrak{L}} & \longrightarrow & \mathfrak{L} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The upper block of the above diagram is the upper block of Diagram (3.2.1) and $\tilde{\mathfrak{L}} := \mathfrak{N}/\mathfrak{L}'^{\star}$. We easily check that all rows and columns are short exact and then the map $\tilde{\mathfrak{L}} \to \mathfrak{L}$ is indeed an isomorphism. One easily checks that the above commutative diagram corresponds to the following commutative diagram of Galois representations

$$
(3.2.5)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Lambda' & \longrightarrow & \Lambda'^{\star} & \xrightarrow{q'} & T' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & {\scriptstyle \wr}\big\uparrow & & \\
0 & \longrightarrow & \Lambda\oplus\Lambda' & \xrightarrow{\tilde{j}} & N & \xrightarrow{\tilde{q}} & T' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & \tilde{\Lambda} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $\tilde{\Lambda}$ is the kernel of map $N \to \Lambda'^{\star}$. We easily see that all rows and all columns are exact and the map $\Lambda \to \tilde{\Lambda}$ is an isomorphism of $\mathbb{Z}_p[G]$-modules. By the construction of filtration in the previous subsection, Diagram (3.2.4) yields a new commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Fil}^i L'^{\star}_K & \longrightarrow & \mathrm{Fil}^i L'_K & \xrightarrow{q'_K} & \mathrm{Fil}^i M'_{K,j'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_K} & & \\
0 & \longrightarrow & \mathrm{Fil}^i \bar{\mathfrak{N}}_K & \longrightarrow & \mathrm{Fil}^i L_K \oplus \mathrm{Fil}^i L'_K & \xrightarrow{\tilde{q}_K} & \mathrm{Fil}^i M'_{K,\tilde{j}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Fil}^i \hat{L}_K & \longrightarrow & \mathrm{Fil}^i L_K & \longrightarrow & Q_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{3.2.6}
$$

where $\mathrm{Fil}^i \hat{L}_K := \mathrm{Fil}^i \bar{\mathfrak{N}}_K / \mathrm{Fil}^i L'^{\star}_K$ and $Q_i := \mathrm{Fil}^i L_K / \mathrm{Fil}^i \hat{L}_K$, which can easily be checked to be $\mathrm{Fil}^i M'_{K,\tilde{j}} / \mathrm{Fil}^i M'_{K,j'}$. We need to show that $p^{c_5} Q_i = 0$ for a constant c_5. Note that $\mathrm{Fil}^i \hat{L}_K$ can be regarded as $h_K(\mathrm{Fil}^i \bar{\mathfrak{N}}_K)$ where $h_K : \bar{\mathfrak{N}}_K \to L_K$ is a surjective map induced by the surjective map of Kisin modules $\mathfrak{N} \to \tilde{\mathfrak{L}} \simeq \mathfrak{L}$. The fact that there exists a constant c_5 such that p^{c_5} kills Q_i is proved in Lemma 3.3.1 in the next subsection.

3.3. **A lemma that needs all constants.** Let $T_{\mathfrak{S}}(\mathfrak{h}) : \Lambda' \to \Lambda$ be a map in $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p}$ and $\mathfrak{h} : \mathfrak{L} \to \mathfrak{L}'$ the corresponding map of Kisin modules. Assume that $\mathfrak{h}$ is surjective. Then we have a surjective map of $\mathcal{O}_K$-modules $h_K : L_K \twoheadrightarrow L'_K$. Note that $h_K : \mathrm{Fil}^i D_K \twoheadrightarrow \mathrm{Fil}^i D'_K$ is surjective, we easily see that $h_K(\mathrm{Fil}^i L_K) \subset \mathrm{Fil}^i L'_K$.

The aim of this subsection is to prove the following lemma:

Lemma 3.3.1. *There exists a constant c_5 only depending on $E(u)$ and r such that $p^{c_5} \mathrm{Fil}^i L'_K \subset h_K(\mathrm{Fil}^i L_K)$.*

We need some preparations for the above lemma. Apply the functor $\mathcal{M}_S$ to $\mathfrak{h}$, we obtain a surjection $\mathfrak{h}_S : \mathcal{L} \twoheadrightarrow \mathcal{L}'$ where $\mathcal{L} := \mathcal{M}_S(\mathfrak{L})$ and $\mathcal{L}' = \mathcal{M}_S(\mathfrak{L}')$ respectively. By the definition in Formula (2.3.1), it is easy to see that $\mathfrak{h}_S(\mathcal{F}^i \mathcal{L}) \subset \mathcal{F}^i \mathcal{L}'$.

Lemma 3.3.2. *Notations as the above. There exists a constant α only depending on r such that $p^{\alpha} \mathcal{F}^r \mathcal{L}' \subset \mathfrak{h}_S(\mathcal{F}^r \mathcal{L})$.*

Proof. Let $\mathcal{K}$ be the kernel of $\mathfrak{h}$. It is not hard to show that $\mathcal{K}$ is a φ-module of $E(u)$-height r (see Proposition 1.3.5 in [Fon90]). In fact $\mathcal{K}$ can be shown to be $\mathfrak{S}$-free but we do not need this here. Now we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathcal{K} & \longrightarrow & \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{L} & \xrightarrow{\mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{h}} & \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{L}' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\varphi} & & \downarrow{\scriptstyle 1\otimes\varphi} & & \downarrow{\scriptstyle 1\otimes\varphi} & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathfrak{L} & \xrightarrow{\mathfrak{h}} & \mathfrak{L}' & \longrightarrow & 0.
\end{array}
$$

It is easy to see both rows are exact as $\mathfrak{L}'$ is finite $\mathfrak{S}$-free. Denote $\mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{L}$, $\mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{L}'$ and $\mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{h}$ by $\mathfrak{L}^\star$, $\mathfrak{L}'^\star$ and $\mathfrak{h}^\star$ respectively. Set

$$\mathcal{F}^r\mathfrak{L}^\star=\{x\in\mathfrak{L}^\star|(1\otimes\varphi)(x)\in E(u)^r\mathfrak{L}\}$$

and define $\mathcal{F}^r\mathfrak{L}'^\star$ similarly. We first prove that $\mathfrak{h}^\star:\mathcal{F}^r\mathfrak{L}^\star\to\mathcal{F}^r\mathfrak{L}'^\star$ is surjective.

To see this, for any $y=\mathfrak{h}^\star(x)\in\mathcal{F}^r\mathfrak{L}'^\star$ with $x\in\mathfrak{L}^\star$, we have $(1\otimes\varphi)(y)\in E(u)^r\mathfrak{L}'$. So there exists a $z\in\mathcal{K}$ such that $(1\otimes\varphi)(x)+z$ is in $E(u)^r\mathfrak{L}$. Then the fact that $\mathfrak{L}$ has height r implies that $(1\otimes\varphi)(x)+z=(1\otimes\varphi)(w')$ for $w'\in\mathfrak{L}^\star$. So there exists $w\in\mathfrak{L}^\star$ such that $(1\otimes\varphi)(w)=z$. As $1\otimes\varphi$ in the last two columns are easily to see to be injective, w is in the kernel of $\mathfrak{h}^\star$. So $\mathfrak{h}^\star(x-w)=y$ and $x-w$ is in $\mathcal{F}^r\mathfrak{L}^\star$. This proves that $\mathfrak{h}^\star:\mathcal{F}^r\mathfrak{L}^\star\to\mathcal{F}^r\mathfrak{L}'^\star$ is surjective.

Now set $\alpha=v_p((r-1)!)$. For any $s\in S$, note that $s=\sum_i a_i\frac{E(u)^i}{i!}$. So $s=s_0+s_1$ with $s_1\in\mathrm{Fil}^rS$ and $p^\alpha s_0\in W(k)[u]$. Now pick any $y=\mathfrak{h}_S(x)\in\mathcal{F}^r\mathcal{L}'$ with $x\in\mathcal{L}$. Then we can write $y=y_0+y_1$ such that $y_1\in\mathrm{Fil}^rS\mathcal{L}'$ and $p^\alpha y_0\in\mathfrak{L}'^\star\subset\mathcal{L}'$. As $\mathfrak{h}_S$ is surjective, there exists $x_1\in\mathrm{Fil}^rS\mathcal{L}\subset\mathcal{F}^r\mathcal{L}$ such that $\mathfrak{h}_S(x_1)=y_1$. It is easy to see that $p^\alpha y_0\in\mathcal{F}^r\mathfrak{L}'^\star$. Hence there exists $x_0\in\mathcal{F}^r\mathfrak{L}^\star\subset\mathcal{F}^r\mathcal{L}$ such that $\mathfrak{h}_S(x_0)=p^\alpha y_0$. This proves the lemma. □

Proof of Lemma 3.3.1. Note that both $\mathfrak{h}_S(\mathcal{F}^i\mathcal{L})$ and $\mathcal{F}^i\mathcal{L}'$ satisfy Griffith Transversality. We denote $\mathrm{F}^i(\mathfrak{h}_S(\mathcal{L}))$ and $\mathrm{F}^i(\mathcal{L}')$ for F^i constructed from $\mathfrak{h}_S(\mathcal{F}^i\mathcal{L})$ and $\mathcal{F}^i\mathcal{L}'$ above Proposition 2.2.1 respectively. Note that the construction of F^i only depends on Fil^r and N on $\mathcal{L}'$. So $p^\alpha\mathrm{F}^i\mathcal{L}'\subset\mathrm{F}^i(\mathfrak{h}_S(\mathcal{L}))\subset\mathrm{F}^i\mathcal{L}'$ by Lemma 3.3.2. Then by Lemma 2.2.1, we get $p^{c_1+\alpha}\mathcal{F}^i\mathcal{L}'\subset p^\alpha\mathrm{F}^i\mathcal{L}'\subset\mathrm{F}^i(\mathfrak{h}_S(\mathcal{L}))\subset\mathfrak{h}_S(\mathcal{F}^i(\mathcal{L}))$. Applying functor f_π to $\mathfrak{h}_S$, we have a surjective map $\tilde h_K:\tilde L_K\twoheadrightarrow\tilde L'_K$ where $\tilde L_K=f_\pi(\mathcal{L})$ and $\tilde L'_K=f_\pi(\mathcal{L}')$. Hence $p^{c_1+\alpha}f_\pi(\mathcal{F}^i\mathcal{L}')\subset f_\pi(\mathfrak{h}_S(\mathcal{F}^i\mathcal{L}))$. By Lemma 2.4.3 , we have

$$p^{c_3+c_1+\alpha}\mathrm{Fil}^i\tilde L'_K\subset p^{c_1+\alpha}f_\pi(\mathcal{F}^i\mathcal{L}')\subset f_\pi(\mathfrak{h}_S(\mathcal{F}^i\mathcal{L}))\subset\tilde h_K(\mathrm{Fil}^i\tilde L_K).$$

Finally, by Proposition 2.4.1 and set $c_5=2c_4+c_3+c_1+\alpha$, we have

$$p^{c_5}\mathrm{Fil}^iL'_K\subset p^{c_4+c_3+c_1+\alpha}\mathrm{Fil}^i\tilde L'_K\subset\tilde h_K(p^{c_4}\mathrm{Fil}^i\tilde L_K)\subset h_K(\mathrm{Fil}^iL_K).$$

□

4. Application to Galois deformation ring

The aim of this section is to reprove a part of Theorem (2.6.7) in [Kis08] via a different approach.

4.1. **p-adic Hodge type.** We first recall the definition of p-adic Hodge type from [Kis08] and prove several technical results on p-adic Hodge type. Let E be a finite extension of $\mathbb{Q}_p$. Suppose that we are given a finite dimensional E-vector space D_E and a filtration $(\mathrm{Fil}^iD_{E,K})_{i\in\mathbb{Z}}$ of $D_{E,K}:=K\otimes_{\mathbb{Q}_p}D_E$ by $E\otimes_{\mathbb{Q}_p}K$-modules such that the associated graded is concentrated in degree in $[0,r]$, namely the set $\{i|\mathrm{gr}^iD_{E,K}\neq0\}\subset\{0,1,\dots,r\}$. We set $\mathbf{v}=\{D_E,\mathrm{Fil}^iD_{E,K},i=0,1,\dots,r\}$. If B is a finite E-algebra and V_B a finite free B-module with a continuous G-action, which makes V_B a de Rham representation, then we say that V_B has *p-adic Hodge type* $\mathbf{v}$ if V_B has all its Hodge-Tate weights in $\{0,\dots,r\}$ and there is an isomorphism of $B\otimes_{\mathbb{Q}_p}K$-modules

$$\mathrm{gr}^i(D_{\mathrm{dR}}(V_B))\simeq\mathrm{gr}^i(D_{E,K})\otimes_EB.$$

Recall that $D_{\mathrm{dR}}(V_B):=(B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}V_B^\vee)^G$.

Lemma 4.1.1. *Notations as the above. Assume that B' is a B-algebra and finite over E. Then $V_{B'} := B' \otimes_B V_B$ has p-adic Hodge type $\mathbf{v}$.*

The above lemma is an easy consequence of the following fact. For any E-algebra A, write $A_K := K \otimes_{\mathbb{Q}_p} A$.

Lemma 4.1.2. (1) *We have $D_{\mathrm{dR}}(V_{B'}) \simeq B' \otimes_B D_{\mathrm{dR}}(V_B)$ and $\mathrm{gr}^i(D_{\mathrm{dR}}(V_{B'})) = B' \otimes_B \mathrm{gr}^i(D_{\mathrm{dR}}(V_B))$ for all $i \in \mathbb{Z}$.*
(2) *$D_{\mathrm{dR}}(V_B)$ is a finite free B_K-module.*

Proof. (1) We first show that $D_{\mathrm{dR}}(V_{B'}) \simeq B' \otimes_B D_{\mathrm{dR}}(V_B)$. Consider the canonical isomorphism $V_B^\vee \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq D_{\mathrm{dR}}(V_B) \otimes_K B_{\mathrm{dR}}$. After tensoring B', we get an isomorphism

$$V_{B'}^\vee \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq B' \otimes_B D_{\mathrm{dR}}(V_B) \otimes_K B_{\mathrm{dR}}.$$

So $\dim_K(B' \otimes_B D_{\mathrm{dR}}(V_B)) = \dim_{\mathbb{Q}_p}(V_{B'})$. But it is obvious that $B' \otimes_B D_{\mathrm{dR}}(V_{B'}) \subset (V_{B'}^\vee \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^G$. Therefore we conclude that

$$B' \otimes_B D_{\mathrm{dR}}(V_B) = D_{\mathrm{dR}}(V_{B'}) = (V_{B'}^\vee \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^G.$$

Similarly, we can show that $B' \otimes_B D_{\mathrm{HT}}(V_B) = D_{\mathrm{HT}}(V_{B'})$, where $D_{\mathrm{HT}}(V) := (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V^\vee)^G$ for a Hodge-Tate representation V.

To show $\mathrm{gr}^i(D_{\mathrm{dR}}(V_{B'})) \simeq B' \otimes_B \mathrm{gr}^i(D_{\mathrm{dR}}(V_B))$ as $B' \otimes_{\mathbb{Q}_p} K$-modules, note that for a de Rham representation V of G, we have $\mathrm{gr}^i(D_{\mathrm{dR}}(V)) \simeq (\mathrm{gr}^i B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^\vee)^G$ for each i and $\bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i(D_{\mathrm{dR}}(V)) \simeq D_{\mathrm{HT}}(V)$. It is clear that $B' \otimes_B \mathrm{gr}^i(D_{\mathrm{dR}}(V_B)) \subset \mathrm{gr}^i(D_{\mathrm{dR}}(V_{B'}))$. Then

$$B' \otimes_B D_{\mathrm{HT}}(V_B) \simeq \bigoplus_{i\in\mathbb{Z}} B' \otimes_B \mathrm{gr}^i(D_{\mathrm{dR}}(V_B)) \subset \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i(D_{\mathrm{dR}}(V_{B'})) \simeq D_{\mathrm{HT}}(V_{B'}).$$

Hence the fact that $B' \otimes_B D_{\mathrm{HT}}(V_B) = D_{\mathrm{HT}}(V_{B'})$ implies that $\mathrm{gr}^i(D_{\mathrm{dR}}(V_{B'})) = B' \otimes_B \mathrm{gr}^i(D_{\mathrm{dR}}(V_B))$.

(2) We have known that if B is a finite extension of E then $D_{\mathrm{dR}}(V_B)$ is a finite free B_K-module (see Lemma 2.1 in [Sav05]). Write $B_{\mathrm{red}} := B/N(B)$ where $N(B)$ is the nilpotent ideal of B. Since B_{red} is a reduced Artinian E-algebra, it is a direct product of finite extension E_j over E for $j = 1, \dots, m$. Hence $D_{\mathrm{dR}}(V_{B_{\mathrm{red}}})$ is a finite free $K \otimes_{\mathbb{Q}_p} B_{\mathrm{red}}$-module where $V_{B_{\mathrm{red}}} = B_{\mathrm{red}} \otimes_B V_B$. By (1), we have $D_{\mathrm{dR}}(V_{B_{\mathrm{red}}}) = B_{\mathrm{red}} \otimes_B D_{\mathrm{dR}}(V_B)$. Let $e_1, \dots, e_d$ be a $K \otimes_{\mathbb{Q}_p} B_{\mathrm{red}}$-basis of $D_{\mathrm{dR}}(V_{B_{\mathrm{red}}})$ with $\hat{e}_i \in D_{\mathrm{dR}}(V_B)$ a lift of e_i and $d = \dim_B(V_B)$. Then by Nakayama's lemma, $\hat{e}_i$ generates $D_{\mathrm{dR}}(V)$ as a B_K-module. Hence there exists a finite free B_K-module M with rank d and a surjection map $f : M \twoheadrightarrow D_{\mathrm{dR}}(V_B)$. On the other hand, it is easy to compute that $\dim_K(D_{\mathrm{dR}}(V_B)) = d \cdot \dim_{\mathbb{Q}_p} B = \mathrm{rank}_K M$. Hence f is an isomorphism and $D_{\mathrm{dR}}(V_B)$ is finite B_K-free. □

Remark 4.1.3. By Remarque 3.1.1.4 in [BM02], $\mathrm{gr}^i D_{\mathrm{dR}}(V_B)$ is not necessarily B_K-free even for $B = E$ being a finite extension of $\mathbb{Q}_p$.

Since $E_K := E \otimes_{\mathbb{Q}_p} K$ is a reduced E-algebra, we have $E_K \simeq \prod_{\iota \in J'} F_{(\iota)}$ of E_K-algebras with $F_{(\iota)}$ a finite extension of E. Here $\iota : K \hookrightarrow F_{(\iota)}$ is an embedding of K to $\overline{\mathbb{Q}}_p$ such that $F_{(\iota)} = E \cdot \iota(K)$ in $\overline{\mathbb{Q}}_p$. So $F_{(\iota)}$ is an E_K-algebra via $\iota : K \hookrightarrow F_{(\iota)}$ and $E \subset F_{(\iota)}$. Hence for any E_K-module M, we get a decomposition $M \simeq \bigoplus_{\iota\in J'} M_{(\iota)}$ with $M_{(\iota)} := F_{(\iota)} \otimes_{E_K} M$. For a filtered E_K-module D_K, we also use $\mathrm{Fil}^i_{(\iota)} D_K$

and $\mathrm{gr}^i_{(\iota)} D_K$ to denote $(\mathrm{Fil}^i D_K)_{(\iota)}$ and $(\mathrm{gr}^i D_K)_{(\iota)}$ respectively. It is easy to check $\mathrm{gr}^i_{(\iota)} D_K \simeq \mathrm{Fil}^i_{(\iota)} D_K/\mathrm{Fil}^{i+1}_{(\iota)} D_K$. Write $B_{F_{(\iota)}} := F_{(\iota)} \otimes_E B$. The following is a useful result:

Lemma 4.1.4. *Notations as the above, V_B has type $\mathbf{v}$ if and only if $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$ is $B_{F_{(\iota)}}$-free and $\mathrm{rank}_{B_{F_{(\iota)}}}(\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))) = \dim_{F_{(\iota)}}(\mathrm{gr}^i_{(\iota)}(D_{E,K}))$ for all $\iota \in J'$ and $i \in \mathbb{Z}$.*

Proof. One direction is clear by definition. Now suppose that $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$ is $B_{F_{(\iota)}}$-free. Select a $B_{F_{(\iota)}}$-basis $e_1, \dots, e_d$ of $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$ and set $M_{(\iota)}$ be $F_{(\iota)}$-module generated by e_i inside $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$. It is obvious that $M_{(\iota)}$ is finite $F_{(\iota)}$-free and $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B)) \simeq B_{F_{(\iota)}} \otimes_{F_{(\iota)}} M_{(\iota)}$. Then $\dim_{F_{(\iota)}} M_{(\iota)} = \dim_{F_{(\iota)}} \mathrm{gr}^i_{(\iota)} D_{E,K}$. Set $M = \bigoplus_{\iota \in J'} M_{(\iota)}$. Then $M \simeq \mathrm{gr}^i D_{E,K}$ as E_K-modules and $B \otimes_E M \simeq \mathrm{gr}^i D_{\mathrm{dR}}(V_B)$ as B_K-modules. □

As the proof of Lemma 4.1.2, write $B_{\mathrm{red}} := B/N(B)$ where $N(B)$ is the nilpotent ideal of B. Since B_{red} is a reduced Artinian E-algebra, it is a direct product of finite extension E_j over E for $j = 1, \dots, m$. Write $V_{E_j} := V_B \otimes_B E_j$ for $j = 1, \dots, m$.

Proposition 4.1.5. *V_B has type $\mathbf{v}$ if and only if V_{E_j} has type $\mathbf{v}$ for each $j = 1, \dots, m$.*

Proof. The "only if" part is the consequence of Lemma 4.1.1. To prove "if" part, note that the fact V_{E_j} has type $\mathbf{v}$ for each $j = 1, \dots, m$ implies that $V_{B_{\mathrm{red}}}$ has type $\mathbf{v}$ where $V_{B_{\mathrm{red}}} := B_{\mathrm{red}} \otimes_B V_B$. So $\mathrm{gr}^i D_{\mathrm{dR}}(V_{B_{\mathrm{red}}}) \simeq B_{\mathrm{red}} \otimes_E \mathrm{gr}^i D_{E,K}$ as $K \otimes_{\mathbb{Q}_p} B_{\mathrm{red}}$-modules. In particular, $\mathrm{gr}^i_{(\iota)} D_{\mathrm{dR}}(V_{B_{\mathrm{red}}})$ is finite $F_{(\iota)} \otimes_E B_{\mathrm{red}}$-free with the rank $d_i = \dim_{F_{(\iota)}} \mathrm{gr}^i_{(\iota)} D_{E,K}$. By Lemma 4.1.4, we have to show that $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$ is $B_{F_{(\iota)}}$-free with rank $\dim_{F_{(\iota)}} \mathrm{gr}^i_{(\iota)} D_{E,K}$. By Lemma 4.1.2 (1), it is easy to check that $\mathrm{gr}^i_{(\iota)} D_{\mathrm{dR}}(V_{B_{\mathrm{red}}}) = B_{\mathrm{red}} \otimes_B \mathrm{gr}^i_{(\iota)} D_{\mathrm{dR}}(V_B)$. Select $e_1, \dots, e_{d_i} \in \mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$ such that the image of $\{e_l\}$ in $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_{B_{\mathrm{red}}}))$ forms a $F_{(\iota)} \otimes_E B_{\mathrm{red}}$-basis of $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_{B_{\mathrm{red}}}))$. By Nakayama's lemma, we know $\{e_l\}$ generates $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$. Hence we have a finite free $B_{F_{(\iota)}}$-module with rank d_i projects $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$. So $\dim_{F_{(\iota)}} \mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B)) \leq d_i \cdot \dim_E B$ and $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$ is finite $B_{F_{(\iota)}}$-free with rank $d_i = \dim_{F_{(\iota)}} \mathrm{gr}^i_{(\iota)} D_{E,K}$ if only if the equality holds. On the other hand, by Lemma 4.1.2 (2), $D_{\mathrm{dR}}(V_B)$ is a finite free B_K-module with rank $d = \mathrm{rank}_B(V_B) = \mathrm{rank}_{B_{\mathrm{red}}}(V_{B_{\mathrm{red}}})$, we have $\dim_{F_{(\iota)}}(D_{\mathrm{dR}}(V_B)_{(\iota)}) = d \dim_E B$. Note that $\dim_{F_{(\iota)}}(D_{E,K,(\iota)}) = d$ because $V_{B_{\mathrm{red}}}$ has type $\mathbf{v}$. Now we have

$$\begin{aligned} d \cdot \dim_E B &= \dim_{F_{(\iota)}}(D_{\mathrm{dR}}(V_B)_{(\iota)}) = \sum_i \dim_{F_{(\iota)}}(\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))) \\ &\leq \sum_i d_i \dim_E B = \sum_i \dim_{F_{(\iota)}}(\mathrm{gr}^i_{(\iota)} D_{E,K}) \dim_E B \\ &= \dim_{F_{(\iota)}} D_{E,K,(\iota)} \dim_E B = d \cdot \dim_E B. \end{aligned}$$

Therefore $\dim_{F_{(\iota)}} \mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B)) = d_i \cdot \dim_E B$ and $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$ is finite $B_{F_{(\iota)}}$-free with rank d_i. This proves the lemma.

□

It will be technically easier to deal with p-adic Hodge type if E contains the Galois closure of K (see the discussion of the next subsection). So we would like to consider the base change V_B to a larger coefficient field. Now let L be a finite extension of E such that L contains the Galois closure of K. We write $B_L := L \otimes_E B$. Obviously, B_L is a finite over L as an L-module. Set $\mathbf{v}' = \{D_L := L \otimes_E D_E, \mathrm{Fil}^i D_{L,K} := L \otimes_E \mathrm{Fil}^i D_{E,K},\ i \in \mathbb{Z}\}$.

Lemma 4.1.6. *Notations as the above. Then V_B has type $\mathbf{v}$ if and only if $V_{B_L} := B_L \otimes_B V_B$ has type $\mathbf{v}'$.*

Proof. By Lemma 4.1.2, it suffices to show that if V_{B_L} has type $\mathbf{v}'$ then V_B has type $\mathbf{v}$. As B_{red} is a product of finite extension of E. $L \otimes_E B_{\mathrm{red}}$ is reduced and then $B_{L,\mathrm{red}} = L \otimes_E B_{\mathrm{red}}$. By Lemma 4.1.5, we can assume that B is a field. Since V_{B_L} has type $\mathbf{v}'$ and $\mathrm{gr}^i(D_{\mathrm{dR}}(V_{B_L})) = L \otimes_E \mathrm{gr}^i(D_{\mathrm{dR}}(V_B))$ by Lemma 4.1.2, we get $L \otimes_E \mathrm{gr}^i(D_{\mathrm{dR}}(V_B)) \simeq B_L \otimes_L \mathrm{gr}^i D_{L,K} = B_L \otimes_E \mathrm{gr}^i D_{E,K}$. Then for each $\iota \in J'$, we have $L \otimes_E \mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B)) \simeq B_L \otimes_E \mathrm{gr}^i_{(\iota)} D_{E,K}$. By Lemma 4.1.4, we have to show that $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$ is finite $B_{F_{(\iota)}}$-free with the rank $= \dim_{F_{(\iota)}}(\mathrm{gr}^i_{(\iota)} D_{E,K})$. Note that $B_L \otimes_E \mathrm{gr}^i_{(\iota)} D_{E,K}$ is finite $B_L \otimes_E F_{(\iota)}$-free with the rank $= \dim_{F_{(\iota)}}(\mathrm{gr}^i_{(\iota)} D_{E,K})$. So it suffices to show that $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_B))$ is finite $B_{F_{(\iota)}}$-free which is the consequence of the next lemma. □

Lemma 4.1.7. *Assume that A, E' are finite extensions of E and B, C are finite extensions of E'. A $B \otimes_E A$-module M is finite $B \otimes_E A$-free if and only if $C \otimes_{E'} M$ is finite $C \otimes_{E'} B \otimes_E A$-free.*

Proof. An easy exercise of counting dimensions. □

4.2. **Filtration of torsion representations with coefficients.** In this subsection, let us first assume that E contains the Galois closure of K (until Theorem 4.2.8). Let $J := \{\iota : K \to \overline{\mathbb{Q}}_p\}$ be the set of all embeddings of K to $\overline{\mathbb{Q}}_p$ and A be a finite flat $\mathcal{O}_E$-algebra. Write $A_{\mathcal{O}_K} := \mathcal{O}_K \otimes_{\mathbb{Z}_p} A$ and $A_K := K \otimes_{\mathbb{Z}_p} A$. For any $\iota \in J$, we have a natural surjection $q_\iota : A_{\mathcal{O}_K} \to A_{(\iota)} := \mathcal{O}_K \otimes_{\mathcal{O}_K,\iota} A$ via $\sum_i a_i \otimes b_i \mapsto \sum_i a_i \iota(b_i)$. Write $q := \bigoplus_{\iota \in J} q_\iota : A_{\mathcal{O}_K} \to \bigoplus_{\iota \in J} A_{(\iota)}$ and $c_\Delta = v_p(\Delta_{K/\mathbb{Q}_p})$ where $\Delta_{K/\mathbb{Q}_p}$ is the discriminant of K over $\mathbb{Q}_p$.

Lemma 4.2.1. *The $\mathcal{O}_K$-algebra map q is injective. Furthermore*
$$p^{c_\Delta}\Big(\bigoplus_{\iota \in J} A_{(\iota)}\Big) \subset q(A_{\mathcal{O}_K}).$$

Proof. As a finite free $\mathbb{Z}_p$-module $\mathcal{O}_K$, select a $\mathbb{Z}_p$-basis $\{1, \alpha, \dots, \alpha^{d-1}\}$ with $d = [K : \mathbb{Q}_p]$. In particular, for any $x \in A_{\mathcal{O}_K}$, x can be written as $\sum\limits_{i=0}^{d-1} a_i \otimes \alpha^i$ with $a_i \in A$. Then
$$q(x) = \left(\sum_{i=0}^{d-1} a_i \iota_0(\alpha^i), \cdots, \sum_{i=0}^{d-1} a_i \iota_{d-1}(\alpha^i)\right)$$
with $\iota_0, \dots, \iota_{d-1}$ running through J. The statement of the lemma follows the fact that the determinant of the matrix $(\iota_m(\alpha^n))_{m,n=0,\dots,d-1}$ is $\Delta_{K/\mathbb{Q}_p} \neq 0$. □

For any $A_{\mathcal{O}_K}$-module M, we write $M_{(\iota)} := M \otimes_{A_{\mathcal{O}_K}, q_\iota} A_{(\iota)}$.

Corollary 4.2.2. *There exists $A_{\mathcal{O}_K}$-modules maps $q_M : M \to \bigoplus_{\iota\in J} M_{(\iota)}$ and $s_M : \bigoplus_{\iota\in J} M_{(\iota)} \to M$ such that $s_M \circ q_M = p^{c_\Delta}\mathrm{Id}_M$ and $q_M \circ s_M = p^{c_\Delta}\mathrm{Id}_{M'}$ where $M' = \bigoplus_{\iota\in J} M_{(\iota)}$.*

In particular, we have the canonical isomorphism $A_K \simeq \prod_{\iota\in J} A_{(\iota)}[\frac{1}{p}]$ as in the previous subsection. If M is an A_K-module then we have a natural decomposition $M = \bigoplus_{\iota\in J} M_{(\iota)}$.

Let $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p,A}$ denote the category whose objects are $A[G]$-modules and also objects in $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p}$. The morphisms in $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p,A}$ are morphisms of $A[G]$-modules. Let L be an object in $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p,A}$. Then by the construction of M_{st}, it is easy to see that $M := M_{\mathrm{st}}(L)$ is a natural $A\otimes_{\mathbb{Z}_p} W(k)$-module. Consequently, $M_K := \mathcal{O}_K \otimes_{\mathbb{Z}_p} M_{\mathrm{st}}(L) \subset D_{\mathrm{dR}}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L)$ is an $A_{\mathcal{O}_K}$-module. For each $i = 1, \dots, r$, $\mathrm{Fil}^i M_K$ is also $A_{\mathcal{O}_K}$-module. We write $\mathrm{Fil}^i_{(\iota)} M_K := A_{(\iota)} \otimes_{q_\iota, A_{\mathcal{O}_K}} \mathrm{Fil}^i M_K$ and $\mathrm{gr}^i_{(\iota)} M_K := A_{(\iota)} \otimes_{q_\iota, A_{\mathcal{O}_K}} \mathrm{gr}^i M_K$. Note we have the right exact sequence

$$\mathrm{Fil}^{i+1}_{(\iota)} M_K \to \mathrm{Fil}^i_{(\iota)} M_K \to \mathrm{gr}^i_{(\iota)} M_K \to 0. \tag{4.2.1}$$

After tensoring $\mathbb{Q}_p$, we obtain an exact sequence:

$$0 \to \mathrm{Fil}^i_{(\iota)} D_K \to \mathrm{Fil}^i_{(\iota)} D_K \to \mathrm{gr}^i_{(\iota)} D_K \to 0.$$

where $D_K := D_{\mathrm{dR}}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L)$. So in particular, we have $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{gr}^i_{(\iota)} M_K \simeq \mathrm{gr}^i_{(\iota)} D_K$. In general it is not clear that the sequence (4.2.1) is left exact, or equivalently, $\mathrm{Fil}^i_{(\iota)} M_K$ is torsion-free (note that $\mathrm{Fil}^i M_K$ in general is not A_K-free, see Remark 4.1.3). However, we can control the torsion part as the following lemma.

Lemma 4.2.3. *Suppose that M is torsion free. Then torsion part of $M_{(\iota)}$ is killed by p^{c_Δ} for any $\iota \in J$*

Proof. Suppose that x is a torsion element in $\bigoplus_{\iota\in J} M_{(\iota)}$. Then $s_M(x)$ is a torsion point in M, which is a torsion free module. So $s_M(x) = 0$. Then by Corollary 4.2.2, $p^{c_\Delta} x = q_M \circ s_M(x) = 0$.

□

Now let us consider the situation of torsion representations. Let $\mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor},A}$ denote the category whose objects are p-power torsion $A[G]$-modules T such that there exists an injective morphism $j : \Lambda_1 \hookrightarrow \Lambda_2$ in $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p,A}$ such that $T \simeq \Lambda_2/j(\Lambda_1)$ as $A[G]$-modules. For a $T \in \mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor},A}$, by the construction of previous section, the exact sequence $0 \to \Lambda_1 \to \Lambda_2 \to T \to 0$ induced by j induces the following exact sequence of $A_{\mathcal{O}_K}$-modules (cf. Corollary 3.1.1)

$$0 \to \mathrm{Fil}^i M_{\mathrm{st}}(\Lambda_2)_K \to \mathrm{Fil}^i M_{\mathrm{st}}(\Lambda_1)_K \to \mathrm{Fil}^i M_{\mathrm{st},j}(T)_K \to 0$$

and

$$0 \to \mathrm{gr}^i M_{\mathrm{st}}(\Lambda_2)_K \to \mathrm{gr}^i M_{\mathrm{st}}(\Lambda_1)_K \to \mathrm{gr}^i M_{\mathrm{st},j}(T)_K \to 0.$$

By tensoring $A_{(\iota)}$ via $q_\iota : A_{\mathcal{O}_K} \to A_{(\iota)}$, we obtain right exact sequences

$$\mathrm{Fil}^i_{(\iota)} M_{\mathrm{st}}(\Lambda_2)_K \to \mathrm{Fil}^i_{(\iota)} M_{\mathrm{st}}(\Lambda_1)_K \to \mathrm{Fil}^i_{(\iota)} M_{\mathrm{st},j}(T)_K \to 0$$

and

$$\mathrm{gr}^i_{(\iota)} M_{\mathrm{st}}(\Lambda_2)_K \to \mathrm{gr}^i_{(\iota)} M_{\mathrm{st}}(\Lambda_1)_K \to \mathrm{gr}^i_{(\iota)} M_{\mathrm{st},j}(T)_K \to 0. \tag{4.2.2}$$

We do not know in general if the sequences are left exact. Suppose that $j' : \Lambda'_1 \hookrightarrow \Lambda'_2$ inside $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p,A}$ is another lift of T then we obtain $\mathrm{Fil}^i M_{\mathrm{st},j'}(T)_K$, $\mathrm{gr}^i M_{\mathrm{st},j'}(T)_K$, $\mathrm{Fil}^i_{(\iota)} M_{\mathrm{st},j'}(T)_K$ and $\mathrm{gr}^i_{(\iota)} M_{\mathrm{st},j'}(T)_K$. By Corollary 2.1.4, there exist morphisms $\tilde g : M_{\mathrm{st},j}(T) \to M_{\mathrm{st},j'}(T)$ and $\tilde g' : M_{\mathrm{st},j'}(T) \to M_{\mathrm{st},j}(T)$ in $\mathrm{M}^r_{\mathrm{tor}}(\varphi, N, \mathrm{Fil})$ such that $\tilde g \circ \tilde g' = p^c \mathrm{Id}|_{M_{\mathrm{st},j'}(T)}$ and $\tilde g' \circ \tilde g = p^c \mathrm{Id}|_{M_{\mathrm{st},j}(T)}$.

Lemma 4.2.4. *$\tilde g$ and $\tilde g'$ are morphisms of $A \otimes_{\mathbb{Z}_p} W(k)$-modules.*

Proof. Note that $\tilde g = p^\beta g$ and $\tilde g' = p^\beta g'$ with a constant $\beta = c_5$ from the last section, where g and g' are morphisms constructed in Corollary 3.1.2 in [Liu11]. It suffices to show that g and g' are morphisms of $A \otimes_{\mathbb{Z}_p} W(k)$-modules, and this has been proved in Proposition 3.4.1 in [Liu11]. □

In particular, $\tilde g$ and $\tilde g'$ induces morphisms $A_{\mathcal{O}_K}$-modules $\tilde g^i : \mathrm{gr}^i M_{\mathrm{st},j}(T)_K \to \mathrm{gr}^i M_{\mathrm{st},j'}(T)_K$ and $\tilde g'^i : \mathrm{gr}^i M_{\mathrm{st},j'}(T)_K \to \mathrm{gr}^i M_{\mathrm{st},j}(T)_K$. By tensoring $A_{(\iota)}$ via $q_\iota : A_{\mathcal{O}_K} \to A_{(\iota)}$ to $\tilde g^i$ and $\tilde g'^i$, we obtain the following result by Corollary 2.1.4.

Corollary 4.2.5. *The maps $\tilde g^i_{(\iota)} : \mathrm{gr}^i_{(\iota)} M_{\mathrm{st},j}(T)_K \to \mathrm{gr}^i_{(\iota)} M_{\mathrm{st},j'}(T)_K$ and $\tilde g'^i_{(\iota)} : \mathrm{gr}^i_{(\iota)} M_{\mathrm{st},j'}(T)_K \to \mathrm{gr}^i_{(\iota)} M_{\mathrm{st},j}(T)_K$ are morphisms of $A_{(\iota)}$-modules and satisfy the following relations:*

$$\tilde g^i_{(\iota)} \circ \tilde g'^i_{(\iota)} = p^c \mathrm{Id}|_{\mathrm{gr}^i_{(\iota)} M_{\mathrm{st},j'}(T)_K} \text{ and } \tilde g'^i_{(\iota)} \circ \tilde g^i_{(\iota)} = p^c \mathrm{Id}|_{\mathrm{gr}^i_{(\iota)} M_{\mathrm{st},j}(T)_K}.$$

We need more preparations to reach the main theorem (Theorem 4.2.8). Let $\Lambda \in \mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p,A}$ such that Λ is a finite free A-module with rank d. Suppose that there exists an ideal $\mathcal{I} \subset A$ such that $A/\mathcal{I} \simeq \mathcal{O}_E/p^n\mathcal{O}_E$ and $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda$ has Hodge type $\mathbf{v}$. Set $d_{(\iota)} := \mathrm{rank}_{B_{(\iota)}}(\mathrm{gr}^0_{(\iota)} D_{\mathrm{dR}}(V))$. Write $M_K := M_{\mathrm{st}}(\Lambda)_K$ as the above.

Lemma 4.2.6. *Assume that A is a local ring and A has a prime ideal $\mathfrak{p}$ such that $A/\mathfrak{p} \simeq \mathcal{O}_F$ with F a finite extension of $\mathbb{Q}_p$. Then $\mathrm{Fil}^0_{(\iota)} M_K$ is finite $A_{(\iota)}$-free with rank d.*

Proof. Let $\mathfrak{M}$ be the Kisin module attached to Λ and $\mathfrak{S}_A = \mathfrak{S} \otimes_{\mathbb{Z}_p} A$. It suffices to show that $\mathfrak{M}$ is a finite free $\mathfrak{S}_A$-module with rank d. Write $A' := A/\mathfrak{p} = \mathcal{O}_F$ and let $\mathfrak{M}_A$, $\mathfrak{M}_{A'}$ be the Kisin modules corresponding to Λ and Λ' respectively. In fact, note that Kisin module is stable under basis change (see the proof of Proposition (1.3) in [Kis08]), we have $\mathfrak{M}_{A'} \simeq A' \otimes_A \mathfrak{M}$ which is indeed a finite free $\mathfrak{S}_{\mathcal{O}_F} := \mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathfrak{S}$-module (see the proof of Proposition (1.6.4) in [Kis08]). Then by Nakayama's lemma, we see that $\mathfrak{M}_A$ is generated by at most d-elements as $\mathfrak{S}_A$-modules where $d = \mathrm{rank}_{\mathfrak{S}_{\mathcal{O}_F}}(\mathfrak{M}_{A'}) = \mathrm{rank}_{A'}(\Lambda') = \mathrm{rank}_A \Lambda$. On the other hand, since Λ is a finite free $\mathbb{Z}_p$-module with rank $[A : \mathbb{Z}_p]d$, we see that $\mathfrak{M}_A$ is a finite free $\mathfrak{S}$-module with rank $[A : \mathbb{Z}_p]d$. Hence $\mathfrak{M}_A$ must be a finite free $\mathfrak{S}_A$-module. □

Lemma 4.2.7. *Assumptions as Lemma 4.2.6. Suppose that $d_{(\iota)} \neq 0$. If $n \geq md+1$ then there exists a $x \in \mathrm{gr}^0_{(\iota)} M_K / \mathcal{I} \mathrm{gr}^0_{(\iota)} M_K$ satisfying $p^m x \neq 0$.*

Proof. For simplicity, we denote $M/\mathcal{I}M$ by $M/\mathcal{I}$ for any A-module M in the following. Let $\tilde{\mathrm{Fil}}^1_{(\iota)}M_K$ be the image of $\mathrm{Fil}^1_{(\iota)}M_K$ inside $\mathrm{Fil}^0_{(\iota)}M_K$ in Equation (4.2.1). By modulo $\mathcal{I}$ to the sequence in Equation (4.2.1), we have sequence

$$\tilde{\mathrm{Fil}}^1_{(\iota)}M_K/\mathcal{I} \to \mathrm{Fil}^0_{(\iota)}M_K/\mathcal{I} \to \mathrm{gr}^0_{(\iota)}M_K/\mathcal{I} \to 0,$$

which is right exact. Write $\bar{M} := \mathrm{Fil}^0_{(\iota)}M_K/\mathcal{I}$ and $\bar{N} \subset \bar{M}$ the submodule of the image of $\tilde{\mathrm{Fil}}^1_{(\iota)}M_K/\mathcal{I}$. We have $\bar{M}/\bar{N} \simeq \mathrm{gr}^0_{(\iota)}M_K/\mathcal{I}$. Now suppose that p^m kills $\bar{M}/\bar{N}$. We would like to derive a contradiction. Since $\bar{M}$ is a finite free $\mathcal{O}_E/p^n\mathcal{O}_E$-module with rank d by the previous lemma, there exists an $\mathcal{O}_E/p^n\mathcal{O}_E$-basis $\bar{e}_1, \dots, \bar{e}_d$ of $\bar{M}$ such that

$$\bar{N} \simeq \mathcal{O}_E/p^n\mathcal{O}_E(\varpi^{a_1}\bar{e}_1) \oplus \cdots \oplus \mathcal{O}_E/p^n\mathcal{O}_E(\varpi^{a_d}\bar{e}_d),$$

where ϖ is a uniformizer of $\mathcal{O}_E$. The statement p^m kills $\bar{M}/\bar{N}$ implies that $\varpi^{a_i}|p^m$ for all $i = 1, \dots, d$. Let $e_1, \dots, e_d$ be a basis of $\mathrm{Fil}^0_{(\iota)}M_K$ which lifts $\bar{e}_1, \dots, \bar{e}_d$ and $y_1, \dots, y_d \in \tilde{\mathrm{Fil}}^1_{(\iota)}M_K$ such that y_i lift $\varpi^{a_i}\bar{e}_i$. then we have $y_i = \varpi^a_i e_i + \sum\limits_{j=1}^{d} b_{ij}e_j$ with $b_{ij} \in \mathcal{I}$. Let X be the $d \times d$-matrix such that $(y_1, \dots, y_d) = (e_1, \dots, e_d)X$. So $\det(X) = \varpi^a + b$ with $a = \sum a_i$ and $b \in \mathcal{I}$. If $n \geq md + 1$ then $\varpi^a|p^{md}$ is not 0 in $A_{(\iota)}/\mathcal{I}$. Hence $\det(X) \neq 0$ in $A_{(\iota)}$. On the other hand, since $\mathrm{gr}^0_{(\iota)}D_{\mathrm{dR}}(V)$ is a finite free $B_{(\iota)}$-module, we can lift a basis $\tilde{z}_1, \dots, \tilde{z}_{d_{(\iota)}}$ of $\mathrm{gr}^0_{(\iota)}D_{\mathrm{dR}}(V)$ to $z_1, \dots, z_{d_{(\iota)}}$ in $\mathrm{Fil}^0_{(\iota)}D_{\mathrm{dR}}(V)$. Since $\det(X)(e_1, \dots, e_d) \subset \mathrm{Fil}^1_{(\iota)}D_{\mathrm{dR}}(V)$, $\det(X)\tilde{z}_i = 0$. This contradicts that $\{\tilde{z}_i\}$ forms a $B_{(\iota)}$-basis of $\mathrm{gr}^0_{(\iota)}D_{\mathrm{dR}}(V)$.

□

In the following, we do not insist that E contains the Galois closure of K.

Theorem 4.2.8. *Let A, A' be finite flat $\mathcal{O}_E$-algebras and $\rho : G \to \mathrm{GL}_d(A)$, $\rho' : G \to \mathrm{GL}_d(A')$ the Galois representations such that $\rho \in \mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p,A}$ and $\rho' \in \mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Z}_p,A'}$ respectively. Suppose that there exist $\mathcal{I} \subset A$ an ideal of A such that $A/\mathcal{I}$ is killed by a power of p, a surjective map $\beta : A' \twoheadrightarrow A/\mathcal{I}$ of $\mathcal{O}_E$-algebras such that $A/\mathcal{I} \otimes_A \rho \simeq \beta' \circ \rho'$ as $A[G]$-modules where $\beta' : \mathrm{GL}_d(A') \twoheadrightarrow \mathrm{GL}_d(A/\mathcal{I})$ is the natural map induced by β.*

Then there exists a constant c^* only depending on K, r and d such that if $\mathcal{I} \subset p^{c^*}A$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \rho'$ has p-adic Hodge type $\mathbf{v}$ then $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \rho$ has type $\mathbf{v}$.

Proof. We first reduce the proof to the situation that $A = \mathcal{O}_E$, A' is local and E contains the Galois closure of K. To see this, write $B := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$ and $B_{\mathrm{red}} := B/N(B)$ with $N(B)$ the nilpotent radical of B. We know that $B_{\mathrm{red}} = \prod_j E_j$ with E_j finite extension of E. Select a Galois extension L such that L contains all Galois closure of E_j and K. Now tensor $\mathcal{O}_L$ via $\mathcal{O}_E$ to $(*)$ and denote $\mathcal{O}_L \otimes_{\mathcal{O}_E} (*)$ by $(*)_{\mathcal{O}_L}$, where $(*)$ is A, A', ρ, ρ', $\mathcal{I}$ and β. Note that $(A_{\mathcal{O}_L}[\frac{1}{p}])_{\mathrm{red}} = L \otimes_E B_{\mathrm{red}} = L \otimes_E \prod_j E_j$. Since L contains the Galois closure of all E_i. So $L \otimes_E \prod_j E_j \simeq \prod_l L$ with E_j embedding to L differently. Let $\psi_l : A_{\mathcal{O}_L} \to (A_{\mathcal{O}_L})[\frac{1}{p}] \twoheadrightarrow L$ be the natural map from $A_{\mathcal{O}_L}$ to l-th factor L of $\prod_l L$. Lemma 4.1.6 and Lemma 4.1.5 imply that it suffices to show that $L \otimes_{\psi_l, A_{\mathcal{O}_L}} \rho$ has type $\mathbf{v}$ [5]. Let $A_l = \psi_l(A_{\mathcal{O}_L})$ and $\mathcal{I}_l = \psi_l(\mathcal{I}_{\mathcal{O}_L})$. It is easy

[5] Strictly speaking, it should be $\mathbf{v}'$ as we has extended the basis field. But it does not matter here by Lemma 4.1.6.

to check that $\mathcal{I}_l \subset p^{c^*} A_l$. Since $\psi_l : A_{\mathcal{O}_L} \twoheadrightarrow A_l \subset L$ is a morphism of $\mathcal{O}_L$-algebras, we see that $A_l = \mathcal{O}_L \subset L$ and obtain a natural projection $\gamma_l : A_{\mathcal{O}_L}/\mathcal{I}_{\mathcal{O}_L} \twoheadrightarrow A_l/\mathcal{I}_l$. Similarly, we can assume that $A'_{\mathcal{O}_L}$ also admits a surjection to $\mathcal{O}_L$. Now replacing $\beta_{\mathcal{O}_L}$ by $\gamma_l \circ \beta_{\mathcal{O}_L}$, $A_{\mathcal{O}_L}$ by A_l, $\mathcal{I}_{\mathcal{O}_L}$ by $\mathcal{I}_l$, A' by $A'_{\mathcal{O}_L}$, E by L respectively and replacing ρ, ρ' accordingly, we can assume that $A = \mathcal{O}_E$, E contains the Galois closure of K and A' admits a surjection to $\mathcal{O}_E$. After localizing A' and β by the maximal ideal containing $\mathrm{Ker}(\beta)$, we can assume that A' is local.

Now we proceed the proof by 2 steps. First step: we prove that there exists a constant c' only depending on K and r such that if $\mathcal{I} \subset p^{c'}\mathcal{O}_E$ then

$$\dim_E \mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V)) \leq \dim_E \mathrm{gr}^i_{(\iota)} D_{\mathrm{dR}}(V').$$

where $V = E \otimes_{\mathcal{O}_E} \rho$ and $V' = E \otimes_{\mathcal{O}_E} \rho'$.

It is easy to see that we can assume that $\mathcal{I} = p^{c'}\mathcal{O}_E$ to complete Step 1. Let T denote the torsion representation $\mathcal{O}_E/p^{c'}\mathcal{O}_E \otimes_{\mathcal{O}_E} \rho \simeq A \otimes_{A/\mathcal{I}} \rho' \in \mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor},\mathcal{O}_E}$. Two lifts ρ and ρ' of T are denoted by j and j' respectively. We write $L_K := M_{\mathrm{st}}(\rho)_K$, $L'_K := M_{\mathrm{st}}(\rho')_K$, $M_K := M_{\mathrm{st},j}(T)_K$ and $M'_K := M_{\mathrm{st},j'}(T)_K$. By the right exact sequence (4.2.2) (and the discussion above (4.2.2)), for each $\iota \in J$, $i \in \mathbb{Z}$ we have $\mathrm{gr}^i_{(\iota)} M_K \simeq \mathrm{gr}^i_{(\iota)} L_K / p^{c'} \mathrm{gr}^i_{(\iota)} L_K$ and $\mathrm{gr}^i_{(\iota)} M'_K \simeq \mathrm{gr}^i_{(\iota)} L'_K / \mathcal{I} \mathrm{gr}^i_{(\iota)} L'_K$. Now Corollary 4.2.5 claims that there exist morphisms of $\mathcal{O}_{E_{(\iota)}}$-modules $\tilde{g}^i_{(\iota)} : \mathrm{gr}^i_{(\iota)} M_K \to \mathrm{gr}^i_{(\iota)} M'_K$, $\tilde{g}'^i_{(\iota)} : \mathrm{gr}^i_{(\iota)} M'_K \to \mathrm{gr}^i_{(\iota)} M_K$ such that $\tilde{g}^i_{(\iota)} \circ \tilde{g}'^i_{(\iota)} = p^c \mathrm{Id}|_{\mathrm{gr}^i_{(\iota)} M'_K}$ and $\tilde{g}'^i_{(\iota)} \circ \tilde{g}^i_{(\iota)} = p^c \mathrm{Id}|_{\mathrm{gr}^i_{(\iota)} M_K}$.

Set $c' := c_\Delta + c + 1$, $d_{(\iota)} := \dim_{E_{(\iota)}}(\mathrm{gr}^i_{(\iota)} D_{\mathrm{dR}}(V))$ and $d'_{(\iota)} := \dim_{E_{(\iota)}}(\mathrm{gr}^i_{(\iota)} D_{\mathrm{dR}}(V'))$. It suffices to show that $d_{(\iota)} \leq d'_{(\iota)}$. As an $\mathcal{O}_E$-module, $\mathrm{gr}^i_{(\iota)} L_K \simeq N_{\mathrm{tor}} + N$ with N_{tor} the torsion part of $\mathrm{gr}^i_{(\iota)} L_K$ and N a finite $\mathcal{O}_E$-free module with rank $d_{(\iota)}$. By Lemma 4.2.3, $\mathrm{gr}^i_{(\iota)} M_K = \mathrm{gr}^i_{(\iota)} L_K / p^{c'} \mathrm{gr}^i_{(\iota)} L_K \simeq N_{\mathrm{tor}} \oplus \bigoplus_{i=1}^{d_{(\iota)}} \mathcal{O}_E / p^{c'} \mathcal{O}_E$. Let $\bar{N}$ denote $p^{c_\Delta} \bigoplus_{i=1}^{d_{(\iota)}} \mathcal{O}_E/p^{c'}\mathcal{O}_E$. By lemma 4.2.3, we get $p^{c_\Delta} \mathrm{gr}^i_{(\iota)} M_K = \bar{N}$. It is clear that $\tilde{g}^i_{(\iota)}(\bar{N}) \subset p^{c_\Delta} \mathrm{gr}^i_{(\iota)} M'_K$ and $\tilde{g}'^i_{(\iota)}(p^{c_\Delta} \mathrm{gr}^i_{(\iota)} M'_K) \subset \bar{N}$. Since $\tilde{g}'^i_{(\iota)} \circ \tilde{g}^i_{(\iota)} = p^c \mathrm{Id}|_{\bar{N}}$ by Corollary 4.2.5, we see that $\tilde{g}'^i_{(\iota)}(\tilde{g}^i_{(\iota)}(\bar{N})) \simeq \bigoplus_{i=1}^{d_{(\iota)}} p^{c+c_\Delta} \mathcal{O}_E / p^{c'} \mathcal{O}_E$, which is a rank $d_{(\iota)}$ finite free $\mathcal{O}_E/p\mathcal{O}_E$-module, is a submodule inside $\tilde{g}'^i_{(\iota)}(p^{c_\Delta} \mathrm{gr}^i_{(\iota)} M'_K)$. Therefore $p^{c_\Delta} \mathrm{gr}^i_{(\iota)} L'_K$ has a surjection to $\tilde{g}'^i_{(\iota)}(p^{c_\Delta} \mathrm{gr}^i_{(\iota)} M'_K)$, which has the following shape

$$\tilde{g}^i_{(\iota)}(p^{c_\Delta} \mathrm{gr}^i_{(\iota)} M'_K) \simeq \mathcal{O}_E/(\varpi^{m_1}) \oplus \mathcal{O}_E/(\varpi^{m_2}) \oplus \cdots \oplus \mathcal{O}_E/(\varpi^{m_{d_{(\iota)}}})$$

with $m_i \geq 1$ and ϖ a uniformizer of E. So the $\mathcal{O}_E$-rank of $p^{c_\Delta} \mathrm{gr}^i_{(\iota)} L'_K$ is at least $d_{(\iota)}$. Finally, by Lemma 4.2.3, we see that the $\mathcal{O}_E$-rank of $p^{c_\Delta} \mathrm{gr}^i_{(\iota)} L'_K$ is just $d'_{(\iota)}$ and we prove that $d_{(\iota)} \leq d'_{(\iota)}$. This completes Step 1. Note that if $d = 1$ then Step 1 implies that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \rho$ does has type $\mathbf{v}$.

Step 2, we show that there exists a constant $\tilde{c} = \tilde{c}(K, r, d)$ only depending on K, r and d such that if $\mathcal{I} \subset p^{\tilde{c}} A$ and $\mathrm{gr}^0_{(\iota)} D_{\mathrm{dR}}(V') \neq \{0\}$ then $\mathrm{gr}^0_{(\iota)} D_{\mathrm{dR}}(V) \neq \{0\}$. In fact, suppose that $\mathrm{gr}^0_{(\iota)}(D_{\mathrm{dR}}(V)) = \{0\}$. Then by the construction of $\mathrm{gr}^0_{(\iota)} M_K$ and Lemma 4.2.1, we conclude that $\mathrm{gr}^0_{(\iota)} M_K$ is killed by p^{c_Δ}. Set $\tilde{c} = (c_\Delta + c)d +$

1. By the construction of $\mathrm{gr}^0_{(\iota)} M'_K$ and Lemma 4.2.7, we see that there exists a $x \in \mathrm{gr}^0_{(\iota)} M'_K$ such that $p^{c_\Delta + c_1} x \neq 0$. However, $p^c x = \tilde{g}^0_{(\iota)}(\tilde{g}'^0_{(\iota)}(x))$ implies that $p^{c_\Delta + c} x = \tilde{g}^0_{(\iota)}(p^{c_\Delta} \tilde{g}'^0_{(\iota)}(x)) = 0$. Contradiction!

Finally, we set $c^* = \tilde{c}(K, dr, d)$ and suppose that $\mathcal{I} \subset p^{c^*} \mathcal{O}_E$. By Step 1 and Step 2, we have seen that $\mathrm{gr}^0_{(\iota)}(D_{\mathrm{dR}}(V)) \neq 0$ if and only if $\mathrm{gr}^0_{(\iota)}(D_{\mathrm{dR}}(V')) \neq 0$. Without loss of generality, we may assume that both $\mathrm{gr}^0_{(\iota)}(D_{\mathrm{dR}}(V))$ and $\mathrm{gr}^0_{(\iota)}(D_{\mathrm{dR}}(V'))$ are nonzero. Now we claim that

$$\dim_{E_{(\iota)}} \mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V)) \leq \mathrm{rank}_{B'_{(\iota)}} \mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V')),$$

where $B'_{(\iota)} = K \otimes_{\iota, K} A'[\frac{1}{p}]$. Note the claim implies the theorem: This is because the claim first implies that

$$\sum_i i \cdot \dim_{E_{(\iota)}} \mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V)) \leq \sum_i i \cdot \mathrm{rank}_{B'_{(\iota)}} \mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V')). \tag{4.2.3}$$

Since the theorem is proved for $\det(\rho)$ and $\det(\rho')$ by the result of Step 1, we conclude that the inequality in Formula (4.2.3) is an equality, and then $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \rho$ has type $\mathbf{v}$. Now it remains to settle the claim. Assume that $i_{(\iota)}$ is the least integer that the claim fails. Then

$$s_{(\iota)} := \sum_{i \leq i_{(\iota)}} i \cdot \mathrm{rank}_{E_{(\iota)}}(\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V)) > t_{(\iota)} := \sum_{i \leq i_{(\iota)}} i \cdot \mathrm{rank}_{B'_{(\iota)}}(\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V')).$$

Note that $s_{(\iota)}$ is the least number that $\dim_{E_{(\iota)}}(\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(\bigwedge^{i_{(\iota)}} V))) \neq 0$ and $t_{(\iota)}$ is the least number that $\mathrm{rank}_{B'_{(\iota)}}(\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(\bigwedge^{i_{(\iota)}} V'))) \neq 0$. But this contradicts to Step 2 and then the claim is proved. □

Remark 4.2.9. Step 1 proved Theorem 1.0.1 and the constant c' does not depend on d. It is natural to ask if c^* can be chosen such that c^* is independent on d. But we do not know the answer.

4.3. **Construction of a certain Galois deformation ring.** Throughout this subsection we fix a p-adic Hodge type $\mathbf{v}$ as the previous subsections. Fix $\mathbb{F}$ a finite extension of $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ and a residue representation $V_{\mathbb{F}} : G \to \mathrm{GL}_n(\mathbb{F})$. Let $\mathscr{C}^0$ be the category whose objects are complete Artinian local rings with residue field $\mathbb{F}$. Morphisms in $\mathscr{C}^0$ are local homomorphisms that are identity on the residue field. Let A be in $\mathscr{C}^0$, $\mathfrak{m}_A$ the maximal ideal and $\Gamma_n(A)$ the kernel of reduction map $q_A : \mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F})$. A homomorphism $V_A : G \to \mathrm{GL}_n(A)$ is called a *lift* (of $V_{\mathbb{F}}$) to A if $q_A \circ V_A = V_{\mathbb{F}}$. We call V_A and V'_A are *strictly equivalent* if $V_A = Y V'_A Y^{-1}$ for some $Y \in \Gamma_n(A)$. The strictly equivalent class of lifts of $V_{\mathbb{F}}$ to A is called a *deformation* of $V_{\mathbb{F}}$ to A. Define a functor $D : \mathscr{C}^0 \to \mathbf{Sets}$ by $D(A) := \{\text{deformations of } V_{\mathbb{F}} \text{ to } A\}$. It is a classical result of Mazur that D is pro-representable by the universal deformation ring $R_{V_{\mathbb{F}}}$ under some suitable hypotheses on $V_{\mathbb{F}}$. In this subsection, we concern the pro-representability of subfunctors of D whose deformations comes from representations satisfying some p-adic Hodge conditions. A lift V_A is called *has type* $\mathbf{v}$ if

- there exists a finite flat $\mathcal{O}_E$-algebra B, a surjective morphism $f : B \to A$ of $\mathcal{O}_E$-algebras and a continuous G-representation on a finite free B-module V_B such that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} V_B$ is semi-stable with p-adic Hodge type $\mathbf{v}$ and $V_B \otimes_f A$ is strictly equivalent to V_A.

Now we consider the following assignment $D^{\mathbf{v}} : \mathscr{C}^0 \to \mathbf{Sets}$ via $D^{\mathbf{v}}(A) = \{$deformations of $V_{\mathbb{F}}$ to A such that a lift of the deformation has type $\mathbf{v}\}$. One has to show that $D^{\mathbf{v}}$ is a functor before to show it is pro-representable.

Lemma 4.3.1. *Notations as the above, $D^{\mathbf{v}}$ is a functor.*

Proof. Let R and S be objects in $\mathscr{C}^0$ and $\phi : R \to S$ be a morphism in $\mathscr{C}^0$. Then it suffices to show that $\rho \in D^{\mathbf{v}}(R)$ implies $\phi \circ \rho \in D^{\mathbf{v}}(S)$. Note that S is a finite R-module (via homomorphism). So there exist a surjective ring homomorphism $\phi' : R[x_1, \dots, x_n] \twoheadrightarrow S$ which extends ϕ and the image of x_i are in the maximal ideal of S. Let I_m denotes the ideal generated by m-th degree homogeneous polynomials. As S is an Artinian ring, $\phi'(I_m) = \{0\}$ for a sufficient large m. So we get a surjection $R[x_1, \dots, x_n]/I_m \twoheadrightarrow S$. Since ρ has type $\mathbf{v}$, there exists a finite flat $\mathcal{O}_E$-algebra B and homomorphism $f : B \to R$ required in the definition. Then f induces a ring homomorphism $B' = B[x_1, \dots, x_n]/I_m \to R[x_1, \dots, x_n]/I_m$ which we still denote by f. Let $V_{B'} = V_B \otimes_B B[x_1, \dots, x_n]/I_m$. It suffices to show that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} V_{B'}$ has type $\mathbf{v}$ but this easily follows Lemma 4.1.1. □

We are going to show that $D^{\mathbf{v}}$ is pro-representable. For this, we need to recall Schlessinger's criteria from [Sch68].

Let $\mathbb{F}[\epsilon] = \mathbb{F}[T]/(T^2)$ with ϵ the image of T. A morphism $R \to S$ in $\mathscr{C}^0$ is called *small* if it is surjective with the kernel a principal ideal which is killed by the maximal ideal of R. Obviously, the natural projection $\mathbb{F}[\epsilon] \to \mathbb{F}$ is small.

Suppose $D : \mathscr{C}^0 \to \mathbf{Sets}$ is a functor satisfying $|D(\mathbb{F})| = 1$. Let the rings $R_0,\ R_1,\ R_2$ and the morphisms $f : R_1 \to R_0$ and $g : R_2 \to R_0$ be in $\mathscr{C}^0$. Consider the natural map

$$(*) \quad D(R_1 \times_{R_0} R_2) \longrightarrow D(R_1) \times_{D(R_0)} D(R_2),$$

where $R_1 \times_{R_0} R_2 := \{(a, b) \in R_1 \times R_2 | f(a) = g(b)\}$ and $D(R_1) \times_{D(R_0)} D(R_2) := \{(a, b) \in D(R_1) \times D(R_2) | D(f)(a) = D(g)(b)\}$. Then Schlessinger's criteria are as follows:

H1 $R_2 \to R_0$ small implies $(*)$ surjective.

H2 If $R_0 = \mathbb{F}, R_2 = \mathbb{F}[\epsilon]$, and $R_2 \to R_0$ is the natural projection then $(*)$ is bijective.

H3 $D(\mathbb{F}[\epsilon])$ a finite-dimensional $\mathbb{F}$-vector space.

H4 If $R_1 = R_2$ and $R_i \to R_0$ $(i = 1, 2)$ are the same small map, then $(*)$ is bijective.

The following results are well-known (see [Sch68] and [Maz89]):

Theorem 4.3.2 (Schlessinger, Mazur)**.**
(1) **H1, H2, H3, H4** *hold if and only if D is pro-representable.*
(2) *Let D be the deformation functor of Galois representations of $V_{\mathbb{F}}$ defined in the beginning of this subsection. If* $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$ *then D is pro-representable.*

In the following, we always assume that D is the deformation functor of Galois representations of $V_{\mathbb{F}}$ defined in the beginning of this subsection. The following is a useful fact, which has been essentially used in [Ram93].

Lemma 4.3.3. *Suppose that D is pro-representable and let D' be a subfunctor of D. Then D' is pro-representable if and only* **H1** *holds for D'.*

Proof. It is easy to check that $D'(R_1 \times_{R_0} R_2)$, $D'(R_1) \times_{D'(R_0)} D'(R_2)$ are subsets of $D(R_1 \times_{R_0} R_2)$, $D(R_1) \times_{D(R_0)} D(R_2)$ respectively. We easily check that **H1** implies **H2** , **H3** and **H4**. □

Proposition 4.3.4. *Suppose that D is pro-representable. Then the deformation functor $D^{\mathbf{v}}$ is pro-representable.*

Proof. By Lemma 4.3.3, we need to prove **H1** holds for $D^{\mathbf{v}}$. Since D is pro-representable, for any $\tilde{\rho} \in D^{\mathbf{v}}(R_1) \times_{D^{\mathbf{v}}(R_0)} D^{\mathbf{v}}(R_2)$. There exists a representation $\rho \in D(R_1 \otimes_{R_0} R_2)$ such that the image ρ of $(*)$ is $\tilde{\rho}$. Note that there exist $\tilde{\rho}_i \in D^{\mathbf{v}}(R_i)$ for $i = 1, 2$ such that $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)$ is in $D^{\mathbf{v}}(R_1) \times_{D^{\mathbf{v}}(R_0)} D^{\mathbf{v}}(R_2)$. Write $R_3 := R_1 \times_{R_0} R_2$. Note that the injection $R_3 \hookrightarrow R_1 \times R_2$ induces an injection of Galois representations $\rho \hookrightarrow \tilde{\rho}_1 \times \tilde{\rho}_2$. Since the strictly equivalent class of $\tilde{\rho}_i$ is in $\in D^{\mathbf{v}}(R_i)$ for each $i = 1, 2$, there exists B_i finite flat $\mathcal{O}_E$-algebras which lift R_i and finite free B_i-representations V_{B_i} which lifts of $\tilde{\rho}_i$ such that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} V_{B_i}$ has type $\mathbf{v}$. Let π be the projection of $B_1 \times B_2$ to $R_1 \times R_2$. Set $B = \{a \in B_1 \times B_2 | \pi(a) \in R_3\} \subset B_1 \times B_2$. It is easy to see that $\pi : B \to R_3$ is a surjective morphism of $\mathcal{O}_E$-algebra and then the continuous group homomorphism $G \to \mathrm{GL}_d(B_2 \times B_2)$ induced by $V_{B_1} \oplus V_{B_2}$ factors through $\mathrm{GL}_d(B)$. So we obtain a Galois representation $V_B : G \to \mathrm{GL}_d(B)$ such that $R_3 \otimes_B V_B \simeq \rho$. It remains to show that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} V_B$ has Hodge type $\mathbf{v}$. Write $C := B[\frac{1}{p}]$ and $C' := (B_1 \times B_2)[\frac{1}{p}]$. Note that C injects in C' and $\mathbb{Q}_p \otimes_{\mathbb{Q}_p} (V_{B_1} \oplus V_{B_2}) \simeq C' \otimes_C (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} V_B)$. We prove that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} V_B$ has type $\mathbf{v}$ via the following lemma. □

Lemma 4.3.5. *Let C' be a finite E-algebra, $C \subset C'$ the E-subalgebra and V_C is a finite free C-module with a continuous G-action which makes V_C a de Rham representation. Then V_C has type $\mathbf{v}$ if and only of $C' \otimes_C V_C$ has type $\mathbf{v}$.*

Proof. Write $V_{C'} := C' \otimes_C V_C$. By Lemma 4.1.1, we only need to show that V_C has type $\mathbf{v}$ if $V_{C'}$ has type $\mathbf{v}$. It is obvious that C_{red} injects C'_{red}. By Lemma 4.1.5 and Lemma 4.1.1, we may assume that both C and C' are fields. By Lemma 4.1.4, we need to show that $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_C))$ is $F_{(\iota)} \otimes_E C$-free with rank $= \dim_{F_{(\iota)}} \mathrm{gr}^i_{(\iota)}(D_{E,K})$ for each $\iota \in J'$. On the other hand, the fact that $V_{C'}$ has type $\mathbf{v}$ implies that $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_{C'})) \simeq C' \otimes_E \mathrm{gr}^i_{(\iota)}(D_{E,K})$ which is finite $F_{(\iota)} \otimes_E C'$-free with the correct rank. As $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_{C'})) \simeq C' \otimes_C (\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_C)))$. Lemma 4.1.7 implies that $\mathrm{gr}^i_{(\iota)}(D_{\mathrm{dR}}(V_C))$ is $F_{(\iota)} \otimes_E C$-free with rank $= \dim_{F_{(\iota)}} \mathrm{gr}^i_{(\iota)}(D_{E,K})$. □

Here comes our main result of this paper:

Theorem 4.3.6. *Assume that the deformation functor D is pro-representable by the ring $R_{V_{\mathbb{F}}}$. Then the subfunctor $D^{\mathbf{v}}$ is pro-representable by a quotient $R^{\mathbf{v}}_{V_{\mathbb{F}}}$ of $R_{V_{\mathbb{F}}}$. Let B be a finite E-algebra and $x : R_{V_{\mathbb{F}}}[\frac{1}{p}] \to B$ be a homomorphism of E-algebras. Then x is semi-stable and has p-adic Hodge type $\mathbf{v}$ if and only if x factors through $R^{\mathbf{v}}_{V_{\mathbb{F}}}$.*

Proof. Let $A := x(R_{V_{\mathbb{F}}})$. Then A is a local finite flat $\mathcal{O}_E$-algebra. If x is semi-stable and has p-adic Hodge type $\mathbf{v}$ then $A/p^n A \otimes_A x$ is in $D^{\mathbf{v}}(A/p^n A)$ for all n. So x factors through $R^{\mathbf{v}}_{V_{\mathbb{F}}}$. Now suppose x factors through $R^{\mathbf{v}}_{V_{\mathbb{F}}}$. Then $A/p^n A \otimes_A x$ is in $D^{\mathbf{v}}(A/p^n A)$ for all n. By the definition of $D^{\mathbf{v}}$, Theorem 4.2.8 and the main theorem in [Liu07], we see that x is semi-stable and has p-adic Hodge type $\mathbf{v}$. □

Remark 4.3.7. The above theorem recovers a part of Theorem (2.6.7) in [Kis08], where the quotient of the universal deformation ring also parameterize potentially semi-stable representation with fixed Galois type τ. Our construction seems more natural as we construct a subfunctor of the deformation functor. It also seems promising that one can fully recover Kisin's theorem if we further require the element in $D^{\mathrm{v}}(A)$ consisting the deformation such that the lift of the deformation are potentially semi-stable and has Galois type τ. But we decide not to study the refined result because we can not see any further advantage of our construction.

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