A NOTE ON LATTICES IN SEMI-STABLE REPRESENTATIONS

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ABSTRACT. Let p be a prime, K a finite extension over $\mathbb{Q}_p$ and $G := \mathrm{Gal}(\overline{K}/K)$. We extend Kisin's theory on φ-modules of finite $E(u)$-height to give a new classification of G-stable $\mathbb{Z}_p$-lattices in semi-stable representations.

[1]

CONTENTS

1. INTRODUCTION

This note serves as a new idea to classify lattices in semi-stable representations. Let k be a perfect field of characteristic p, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[\frac{1}{p}]$, K/K_0 a finite totally ramified extension, $\overline{K}$ a fixed algebraic closure of K and $G := \mathrm{Gal}(\overline{K}/K)$. For many technical reasons, we are interested in classifying G-stable $\mathbb{Z}_p$-lattices in semi-stable p-adic Galois representations, via linear algebra data like admissible filtered (φ, N)-modules in Fontaine's theory. Many important steps have been made in this direction. Examples include Fontaine

1991 *Mathematics Subject Classification.* Primary 14F30,14L05.
Key words and phrases. semi-stable representations, Kisin modules, G-stable $\mathbb{Z}_p$-lattices .
[1]This is the 4th draft.

and Laffaille' theory [FL82] on strongly divisible $W(k)$-lattices in filtered (φ, N)-modules, Breuil's theory on strongly divisible S-lattices ([Bre02], [Liu08b]), and Wach and Berger's theory on Wach modules ([Wac96], [Ber04]). Unfortunately, these classifications always have some restrictions (on the absolute ramification index, Hodge-Tate weights, etc). Based on Kisin's theory in [Kis06], the aim of this paper is to provide a classification without these restrictions.

More precisely, let $E(u)$ be an Eisenstein polynomial for a fixed uniformizer π of K, $K_\infty = \cup_{n\geq 1} K(\sqrt[p^n]{\pi})$, $G_\infty = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\mathfrak{S} = W(k)[\![u]\!]$. We equip $\mathfrak{S}$ with the endomorphism φ which acts via Frobenius on $W(k)$, and sends u to u^p. Let $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the $\mathfrak{S}$-linear map $1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. Objects in $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ are called *φ-modules of $E(u)$-height r* or *Kisin modules*. In [Kis06], Kisin proved that any G_∞-stable $\mathbb{Z}_p$-lattice T in a semi-stable Galois representation comes from a Kisin module (see Theorem 2.1.1 for details). Obviously, extra data have to be added if one would like to extend the classification of G_∞-stable lattices to the classification of G-stable lattices. Our idea is to imitate the theory of (φ, Γ)-modules. But G_∞ is not a normal subgroup of G and there is no apparent natural G-action on $\mathfrak{S}$. To remedy this, we construct an $\mathfrak{S}$-algebra $\widehat{\mathcal{R}}$ inside $W(R)$ (see §2.1 for the construction of $\widehat{\mathcal{R}} \subset W(R)$) such that $\widehat{\mathcal{R}}$ is stable under Frobenius and the G-action. Furthermore, the G-action on $\widehat{\mathcal{R}}$ factors through $\hat{G} := \mathrm{Gal}(K_{\infty, p^\infty}/K)$ where K_{∞, p^∞} is the Galois closure of K_∞ over K. The construction of $\widehat{\mathcal{R}}$ allows us to define a $(\varphi, \hat{G})$-module to be a Kisin module $(\mathfrak{M}, \varphi)$ with an extra semi-linear $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ compatible with Frobenius (see Definition 2.2.3 for details). Our main result in this note is that the category of G-stable $\mathbb{Z}_p$-lattices in semi-stable representations with Hodge-Tate weights in $\{0, \dots, r\}$ is anti-equivalent to the category of $(\varphi, \hat{G})$-modules of $E(u)$-height r.

Just as any integral version of p-adic Hodge theory before, $(\varphi, \hat{G})$-modules will help us better to understand the reduction of semi-stable representations, and this will be discussed in forthcoming work (eg. [CL08], [Liu08a]). On the other hand, so far we do not fully understand the structure of $\widehat{\mathcal{R}}$. In fact, $\widehat{\mathcal{R}}$ seems quite complicated (see Example 3.2.3). So at least at this stage, it seems that our theory only serves as a theoretic approach. We hope we can simplify this theory in the future by further exploring the structure of $\widehat{\mathcal{R}}$, such that we could provide more explicit examples or carry out some concrete computations by $(\varphi, \hat{G})$-modules.

Convention 1.0.1. We define various Frobenius structures on different rings and modules. The symbol φ is reserved to denote Frobenius. We sometime add subscripts to indicate on which object Frobenius is defined. For example, $\varphi_{\mathfrak{M}}$ is the Frobenius defined on $\mathfrak{M}$. We always drop these subscripts if no confusions arise. For any finite free $\mathbb{Z}_p$-module T, we use $T^\vee$ to denote its $\mathbb{Z}_p$-dual $\mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)$. Finally, we denote $\gamma_i(x)$ the standard divided power $\frac{x^i}{i!}$ and Id the identity map.

Acknowledgment: It is a pleasure to thank Xavier Caruso for very useful comments. We also thank the referee for encouraging us to include the case $p = 2$.

2. Preliminary and the Main Result

2.1. **Kisin modules.** Recall that k is a perfect field of characteristic p, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[\frac{1}{p}]$, K/K_0 a finite totally ramified extension and $e = e(K/K_0)$ the absolute ramification index. Throughout this paper we fix a uniformiser $\pi \in K$ with Eisenstein polynomial $E(u)$. Recall that $\mathfrak{S} = W(k)[\![u]\!]$ is equipped with a Frobenius endomorphism φ via $u \mapsto u^p$ and the natural Frobenius on $W(k)$. A *φ-module* (over $\mathfrak{S}$) is an $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi : \mathfrak{M} \to \mathfrak{M}$. A morphism between two φ-modules $(\mathfrak{M}_1, \varphi_1)$, $(\mathfrak{M}_2, \varphi_2)$ is an $\mathfrak{S}$-linear morphism compatible with the φ_i. Denote by $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ the category of *φ-modules of $E(u)$-height r* in the sense that $\mathfrak{M}$ is finite free [2] over $\mathfrak{S}$ and the cokernel of φ^* is killed by $E(u)^r$, where φ^* is the $\mathfrak{S}$-linear map $1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$. Objects in $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ are also called *Kisin modules* (*of height* [3] r).

Let $R = \varprojlim \mathcal{O}_{\overline{K}}/p$ where the transition maps are given by Frobenius. By the universal property of the Witt vectors $W(R)$ of R, there is a unique surjective projection map $\theta : W(R) \to \widehat{\mathcal{O}}_{\overline{K}}$ to the p-adic completion of $\mathcal{O}_{\overline{K}}$ which lifts the projection $R \to \mathcal{O}_{\overline{K}}/p$ onto the first factor in the inverse limit. Let $\pi_n \in \overline{K}$ be a p^n-th root of π, such that $(\pi_{n+1})^p = \pi_n$; write $\underline{\pi} = (\pi_n)_{n \geq 0} \in R$ and let $[\underline{\pi}] \in W(R)$ be the Teichmüller representative. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ by the map $u \mapsto [\underline{\pi}]$, This embedding extends to an embedding $\mathfrak{S} \hookrightarrow W(R)$ which is compatible with Frobenious endomorphisms.

Denote by $\mathcal{O}_{\mathcal{E}}$ the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$. Then $\mathcal{O}_{\mathcal{E}}$ is a discrete valuation ring with residue field the Laurent series ring $k(\!(u)\!)$. We write $\mathcal{E}$ for the field of fractions of $\mathcal{O}_{\mathcal{E}}$. If $\mathrm{Fr}R$ denotes the field of fractions of R, then the inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to an inclusion $\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\mathrm{Fr}R)$. Let $\mathcal{E}^{\mathrm{ur}} \subset W(\mathrm{Fr}R)[\frac{1}{p}]$ denote the maximal unramified extension of $\mathcal{E}$ contained in $W(\mathrm{Fr}R)[\frac{1}{p}]$, and $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ its ring of integers. Since $\mathrm{Fr}R$ can be seen to be algebraically closed (see Théorème A 3.1.6 [Fon90]), the residue field $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ is the separable closure of $k(\!(u)\!)$. We denote by $\widehat{\mathcal{E}^{\mathrm{ur}}}$ the p-adic completion of $\mathcal{E}^{\mathrm{ur}}$, and by $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ its ring of integers. $\widehat{\mathcal{E}^{\mathrm{ur}}}$ is also equal to the closure of $\mathcal{E}^{\mathrm{ur}}$ in $W(\mathrm{Fr}R)[\frac{1}{p}]$. We write $\mathfrak{S}^{\mathrm{ur}} = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \cap W(R) \subset W(\mathrm{Fr}R)$ [4]. We regard all these rings as subrings of $W(\mathrm{Fr}R)[\frac{1}{p}]$.

Recall that $K_\infty = \bigcup_{n \geq 0} K(\pi_n)$ and $G_\infty = \mathrm{Gal}(\overline{K}/K_\infty)$. G_∞ acts continuously on $\mathfrak{S}^{\mathrm{ur}}$ and $\mathcal{E}^{\mathrm{ur}}$ and fixes the subring $\mathfrak{S} \subset W(R)$. Finally, we denote by $\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ the category of continuous $\mathbb{Z}_p$-linear representations of G_∞ on finite free $\mathbb{Z}_p$-modules.

To any Kisin module $(\mathfrak{M}, \varphi)$, one can associate a $\mathbb{Z}_p[G_\infty]$-module:

$$T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}).$$

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[2] This is a somewhat ad hoc definition because we are only concerned with finite free $\mathfrak{S}$-modules here. In fact the definition of φ-modules of finite $E(u)$-height works equally well if one requires only that $\mathfrak{M}$ is of finite type over $\mathfrak{S}$, and this more general notion is useful when one studies p-power torsion representations.

[3] Throughout this paper, the height is always $E(u)$-height. So we always omit "$E(u)$".

[4] The careful reader may notice that our definition of $\mathfrak{S}^{\mathrm{ur}}$ differs slightly from the one in [Kis06]; this is a typo in [Kis06] which has been corrected in (E.3) in [Kis08].

One can show that $T_{\mathfrak{S}}(\mathfrak{M})$ is finite free over $\mathbb{Z}_p$ and $\mathrm{rank}_{\mathbb{Z}_p}(T_{\mathfrak{S}}(\mathfrak{M})) = \mathrm{rank}_{\mathfrak{S}}(\mathfrak{M})$ (see for example, Corollary (2.1.4) in [Kis06]). Let V be a continuous linear representation of $G := \mathrm{Gal}(\overline{K}/K)$ on a finite dimensional $\mathbb{Q}_p$-vector space. V is said to be of *$E(u)$-height r* if there exists a G_∞-stable $\mathbb{Z}_p$-lattice $T \subset V$ and a Kisin module $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that $T \simeq T_{\mathfrak{S}}(\mathfrak{M})$. We refer [Fon94b] to the notion of *semi-stable* p-adic Galois representations. The following theorem summarizes the known results on the relation between semi-stable representations and representations of finite $E(u)$-height.

Theorem 2.1.1 ([Kis06]).
(1) *The functor $T_{\mathfrak{S}} : \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ is fully faithful.*
(2) *A semi-stable representation with Hodge-Tate weights in $\{0,\dots,r\}$ is of finite $E(u)$-height r.*

Remark 2.1.2.
(1) Suppose that V is of $E(u)$-height r. Then it is easy to show that any G_∞-stable $\mathbb{Z}_p$-lattice $T \subset V$ comes from a Kisin module $\mathfrak{N} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$, i.e., $T \simeq T_{\mathfrak{S}}(\mathfrak{N})$. See the proof of Lemma (2.1.15) in [Kis06].
(2) It is natural to ask if the converse question for Theorem 2.1.1 (2) is true. Unfortunately, it is not always true. See Example 4.2.1 and the refined form of this question in the end of §4.

2.2. **$(\varphi, \hat{G})$-modules.** We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. There is a unique continuous map (Frobenius) $\varphi : S \to S$ which extends the Frobenius on $\mathfrak{S}$. Define a continuous $W(k)$-linear derivation $N : S \to S$ such that $N(u) = -u$. We denote $S[1/p]$ by S_{K_0}.

Recall $R = \varprojlim \mathcal{O}_{\overline{K}}/p$ and the unique surjective map $\theta : W(R) \to \widehat{\mathcal{O}}_{\overline{K}}$ which lifts the projection $R \to \mathcal{O}_{\overline{K}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$ with respect to $\mathrm{Ker}(\theta)$ and we naturally extend θ to $\theta : A_{\mathrm{cris}} \to \widehat{\mathcal{O}}_{\overline{K}}$. Recall that $[\underline{\pi}] \in W(R)$ is the Teichmüller representative of $\underline{\pi} = (\pi_n)_{n\geq 0} \in R$ and we embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ via $u \mapsto [\underline{\pi}]$. This embedding extends to an embedding $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$ which is compatible with Frobenius endomorphisms. As usual, we write $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$ and $B^+_{\mathrm{st}} = B^+_{\mathrm{cris}}[\mathfrak{u}]$ with $\mathfrak{u} = \log(u)$.

For any field extension $F/\mathbb{Q}_p$, set $F_{p^\infty} = \bigcup_{n=1}^{\infty} F(\zeta_{p^n})$ with ζ_{p^n} a primitive p^n-th root of unity. Note that $K_{\infty,p^\infty} = \bigcup_{n=1}^{\infty} K(\pi_n, \zeta_{p^n})$ is the Galois closure of K_∞ over K. Set $\hat{K} := K_{\infty,p^\infty}$, $\hat{G} := \mathrm{Gal}(\hat{K}/K)$, $H_K := \mathrm{Gal}(\hat{K}/K_\infty)$ and $G_{p^\infty} := \mathrm{Gal}(\hat{K}/K_{p^\infty})$ [5].

For any $g \in G$, let $\underline{\epsilon}(g) = g(\underline{\pi})/\underline{\pi} = (\epsilon_i(g))_{i\geq 0} \in R^*$ where $\epsilon_i(g)$ is a p^i-th root of unity. Fix a choice of *primitive* p^i-root of unity ζ_{p^i} for $i \geq 0$ and set $\underline{\epsilon} := (\zeta_{p^i})_{i\geq 0} \in R$ and $t := \log([\underline{\epsilon}]) \in A_{\mathrm{cris}}$. We see that $g(t) = \chi(g)t$ with χ the p-adic cyclotomic character, and there exists an $\alpha(g) \in \mathbb{Z}_p$ such that $\log([\underline{\epsilon}(g)]) = \alpha(g)t$.

The projection of R to $\bar{k}$ induces a projection $\nu : W(R) \to W(\bar{k})$. Since $\nu(\mathrm{Ker}(\theta)) = pW(\bar{k})$, the map ν extends naturally to a map $\nu : A_{\mathrm{cris}} \to W(\bar{k})$,

[5] Here we use a different notation from those in [Liu07] and [Liu08b], where we use G_0 to denote G_{p^∞}.

and $\nu : B^+_{\mathrm{cris}} \to W(\bar{k})[\frac{1}{p}]$. Write $I_+B^+_{\mathrm{cris}} = \mathrm{Ker}(\nu)$ and $I_+A := \mathrm{Ker}(\nu) \cap A$ for any subring $A \subset B^+_{\mathrm{cris}}$. Since $\nu(u) = 0$, it is easy to check that $I_+\mathfrak{S} = u\mathfrak{S}$ and $I_+S = \{f = \sum\limits_{i=1}^{\infty} a_i u^i | a_i \in K_0,\ f \in S\}$.

For any integer $n \geq 0$, let $t^{\{n\}} = t^{r(n)}\gamma_{\tilde{q}(n)}(\frac{t^{p-1}}{p}) \in A_{\mathrm{cris}}$ where $n = (p-1)\tilde{q}(n) + r(n)$ with $0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power. Define a subring $\mathcal{R}_{K_0}$ of B^+_{cris} as in §6, [Liu07]:

$$\mathcal{R}_{K_0} = \left\{ \sum_{i=0}^{\infty} f_i t^{\{i\}}, f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to +\infty \right\}.$$

Define $\widehat{\mathcal{R}} := \mathcal{R}_{K_0} \cap W(R)$ and $I_+ := I_+\widehat{\mathcal{R}}$. The following lemma lists some useful facts on $\widehat{\mathcal{R}}$ and $\mathcal{R}_{K_0}$.

Lemma 2.2.1. (1) *$\widehat{\mathcal{R}}$ (resp. $\mathcal{R}_{K_0}$) is a φ-stable $\mathfrak{S}$-algebra as a subring in $W(R)$ (resp. B^+_{cris}).*

(2) *$\widehat{\mathcal{R}}$ and I_+ (resp. $\mathcal{R}_{K_0}$ and $I_+\mathcal{R}_{K_0}$) are G-stable. The G-action on $\widehat{\mathcal{R}}$ and I_+ (resp. $\mathcal{R}_{K_0}$ and $I_+\mathcal{R}_{K_0}$) factors though $\hat{G}$.*

(3) *$\mathcal{R}_{K_0}/I_+\mathcal{R}_{K_0} \simeq K_0$ and $\widehat{\mathcal{R}}/I_+ \simeq S/I_+S \simeq \mathfrak{S}/u\mathfrak{S} \simeq W(k)$.*

Proof. (1) It is obvious because $\varphi(u) = u^p$ and $\varphi(t) = pt$.

(2) We first check that $\mathcal{R}_{K_0}$ is G-stable. For any $g \in G$ since $g(t) = \chi(g)t$, it suffices to check that $g(f) \in \mathcal{R}_{K_0}$ for any $f \in S$. An easy calculation shows that

$$g(f) = \sum_{i=0}^{\infty} N^i(f)\gamma_i(-\log([\underline{\epsilon}(g)])). \tag{2.2.1}$$

Note that $\log([\underline{\epsilon}(g)]) = \alpha(g)t$ for some $\alpha(g) \in \mathbb{Z}_p$ and $\gamma_i(t) \to 0$ in A_{cris} p-adically (see §5.2.4 in [Fon94a]). So we see that $\mathcal{R}_{K_0}$ is G-stable and the G-action on $\mathcal{R}_{K_0}$ factors through $\hat{G}$. Since $W(R)$ is G-stable as a subring of B^+_{cris}, $\widehat{\mathcal{R}}$ is G-stable and the G-action on $\widehat{\mathcal{R}}$ factors through $\hat{G}$. Noting that the map $\nu : B^+_{\mathrm{cris}} \to W(\bar{k})[\frac{1}{p}]$ is G-equivariant, we see that $I_+B^+_{\mathrm{cris}}$, hence $I_+\mathcal{R}_{K_0}$ and I_+ is G-stable.

(3) Note that u and $[\underline{\epsilon}]-1$ are in $I_+W(R)$, so $u,\ t \in I_+B^+_{\mathrm{cris}}$, and hence $\nu(\mathcal{R}_{K_0}) = K_0$. Then $\widehat{\mathcal{R}}/I_+ = \nu(\widehat{\mathcal{R}}) \subset \nu(\mathcal{R}_{K_0}) = K_0$. Note that $\widehat{\mathcal{R}}/I_+ \hookrightarrow \nu(W(R)) = W(\bar{k})$. Thus we get $\widehat{\mathcal{R}}/I_+ \simeq K_0 \cap W(\bar{k}) = W(k)$. □

Remark 2.2.2. By Lemma 7.1.2 in [Liu07], $\mathcal{R}_{K_0} \subset K_0[\![x, y]\!]$ via $u \mapsto x$ and $t \mapsto y$. The structure of $\widehat{\mathcal{R}}$ is much more complicated and so far we do not know how to describe it explicitly. See Example 3.2.3.

Let $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ be a Kisin module of height r and $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Then we can naturally extend φ from $\mathfrak{M}$ to $\hat{\mathfrak{M}}$ by

$$\varphi_{\hat{\mathfrak{M}}}(a \otimes m) = \varphi_{\widehat{\mathcal{R}}}(a) \otimes \varphi_{\mathfrak{M}}(m), \quad \forall a \in \widehat{\mathcal{R}},\ \forall m \in \mathfrak{M}.$$

Definition 2.2.3. A $(\varphi, \hat{G})$-module (of height r) is a triple $(\mathfrak{M}, \varphi, \hat{G})$ where

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Kisin module (of height r),

(2) $\hat{G}$ is an $\widehat{\mathcal{R}}$-semi-linear $\hat{G}$-action on $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$,

(3) $\hat{G}$ commutes with $\varphi_{\hat{\mathfrak{M}}}$ on $\hat{\mathfrak{M}}$, i.e., for any $g \in \hat{G}$, $g\varphi_{\hat{\mathfrak{M}}} = \varphi_{\hat{\mathfrak{M}}}g$,

(4) regarding $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule in $\hat{\mathfrak{M}}$, we have $\mathfrak{M} \subset \hat{\mathfrak{M}}^{H_K}$,
(5) $\hat{G}$ acts on the $W(k)$-module $M := \hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}} \simeq \mathfrak{M}/u\mathfrak{M}$ trivially.

A morphism between two $(\varphi, \hat{G})$-modules is a morphism of Kisin modules that commutes with the $\hat{G}$-action on $\hat{\mathfrak{M}}$'s. We denote by $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$ the category of $(\varphi, \hat{G})$-modules of height r. Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ be a $(\varphi, \hat{G})$-module. In the following, we often abuse the notations by letting $\hat{\mathfrak{M}}$ denote the ambient module $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ if no confusions arise. We always regard $\hat{\mathfrak{M}} = \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ as a G-module via the projection $G \twoheadrightarrow \hat{G}$.

2.3. **The main theorem.** Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ be a $(\varphi, \hat{G})$-module. We can associate a $\mathbb{Z}_p[G]$-module:

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, W(R)), \tag{2.3.1}$$

where G acts on $\hat{T}(\hat{\mathfrak{M}})$ via $g(f)(x) = g(f(g^{-1}(x)))$ for any $g \in G$ and $f \in \hat{T}(\hat{\mathfrak{M}})$. Now we can state our main theorem:

Theorem 2.3.1. (1) *Let $\hat{\mathfrak{M}} := (\mathfrak{M}, \varphi, \hat{G})$ be a $(\varphi, \hat{G})$-module. There is a natural isomorphism of $\mathbb{Z}_p[G_\infty]$-modules*

$$\theta : T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \xrightarrow{\sim} \hat{T}(\hat{\mathfrak{M}}) = \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, W(R)). \tag{2.3.2}$$

(2) *$\hat{T}$ induces an anti-equivalence between the category of $(\varphi, \hat{G})$-modules of height r and the category of G-stable $\mathbb{Z}_p$-lattices in semi-stable representations with Hodge-Tate weights in $\{0, \dots, r\}$.*

3. The Proof of the Main Theorem

3.1. **The connection to Kisin's theory.** In this subsection, we will prove Theorem 2.3.1 (1) and that $\hat{T}$ is well-defined and fully faithful.

Let $(\mathfrak{M}, \varphi, \hat{G})$ be a $(\varphi, \hat{G})$-module and $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. As in Definition 2.2.3, we regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $\hat{\mathfrak{M}}$. Then for any $f \in T_{\mathfrak{S}}(\mathfrak{M})$, define $\theta(f) \in \mathrm{Hom}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}}, W(R))$ by

$$\theta(f)(a \otimes x) := a\varphi(f(x)), \quad \forall a \in \widehat{\mathcal{R}},\ \forall x \in \mathfrak{M}.$$

It is routine to check that $\theta(f)$ is well-defined and preserves Frobenius. Therefore, the $\mathbb{Z}_p$-linear map $\theta : T_{\mathfrak{S}}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})$ is well-defined. Now we have reduced the proof of Theorem 2.3.1 (1) to the following

Lemma 3.1.1. *$\theta : T_{\mathfrak{S}}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})$ is an isomorphism of $\mathbb{Z}_p[G_\infty]$-modules.*

Proof. Since $\varphi : \mathfrak{S}^{\mathrm{ur}} \to W(R)$ is injective, θ is obviously an injection. To see that θ is surjective, for any $h \in \hat{T}(\hat{\mathfrak{M}})$, consider $f := h|_{\mathfrak{M}}$. Since f is a $\varphi(\mathfrak{S})$-linear morphism from $\mathfrak{M}$ to $W(R) = \varphi(W(R))$, there exists an $\mathfrak{f} \in \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, W(R))$ such that $\varphi(\mathfrak{f}) = f$. Obviously, $\theta(\mathfrak{f}) = h$ and $\mathfrak{f}$ preserves Frobenius. Now we have $\mathfrak{f} \in \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W(R))$. It suffices to show that $\mathfrak{f} \in T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})$. Note that $\mathfrak{f}(\mathfrak{M}) \subset W(R)$ is an $\mathfrak{S}$-finite type φ-stable submodule and of $E(u)$-height r. By [Fon90], Proposition B 1.8.3, we have $\mathfrak{f}(\mathfrak{M}) \subset \mathfrak{S}^{\mathrm{ur}}$. This completes the proof of the bijectivity of θ. Now it suffices to check that θ is compatible with the G_∞-actions on both sides. For any $g \in G_\infty$, $a \in \widehat{\mathcal{R}}$, $x \in \mathfrak{M}$ and $f \in$

$T_{\mathfrak{S}}(\mathfrak{M})$, $g(\theta(f))(a\otimes x)=g(\theta(f)(g^{-1}(a\otimes x)))$. Note that G_∞ acts on $\mathfrak{M}$ trivially by Definition 2.2.3 (4), and so we have

$$g(\theta(f)(g^{-1}(a\otimes x)))=g(\theta(f)(g^{-1}(a)\otimes x))=ag(\varphi(f(x)))=\theta(g(f))(a\otimes x).$$

That is, $g(\theta(f))=\theta(g(f))$. $\square$

Now we need some preparations to show that $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\hat{T}(\hat{\mathfrak{M}})$ is semi-stable. Let T be a finite free $\mathbb{Z}_p$-representation of G or G_∞, and denote by $T^\vee$ the $\mathbb{Z}_p$-dual of T. It will be useful to recall the following technical results from [Liu07], §3.2. Let $\mathfrak{M}$ be a Kisin module of height r. Using the definition of $T_{\mathfrak{S}}(\mathfrak{M})$, we can show (cf. [Liu07], Proposition 3.2.1) there exists an $\mathfrak{S}^{\mathrm{ur}}$-linear, G_∞-compatible morphism[6]

$$\iota_{\mathfrak{S}}:\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}\to\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^\vee_{\mathfrak{S}}(\mathfrak{M}).$$

Select a $\mathfrak{t}\in\mathfrak{S}^{\mathrm{ur}}$ such that $\mathfrak{t}\notin p\mathfrak{S}^{\mathrm{ur}}$ and $\varphi(\mathfrak{t})=c_0^{-1}E(u)\mathfrak{t}$ where pc_0 is the constant term of $E(u)$. Such $\mathfrak{t}$ is unique up to units of $\mathbb{Z}_p$, see Example 2.3.5 in [Liu07] for details.

Lemma 3.1.2. *$\iota_{\mathfrak{S}}$ is an injection and $\mathfrak{t}^r(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^\vee_{\mathfrak{S}}(\mathfrak{M}))\subset\iota_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M})$.*

Proof. Theorem 3.2.2 in [Liu07] proved the construction of $\iota_{\mathfrak{S}}$ (cf. the construction of $\hat{\iota}$ in [Liu07]), and that there exists an $\mathfrak{S}^{\mathrm{ur}}$-linear map

$$\iota^\vee_{\mathfrak{S}}:\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^\vee_{\mathfrak{S}}(\mathfrak{M})\to\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}$$

(cf. $\hat{\iota}^\vee$ in [Liu07]) such that $\iota^\vee_{\mathfrak{S}}\circ\iota_{\mathfrak{S}}=\mathfrak{t}^r\mathrm{Id}$ where Id is the identity map. Note that both $\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^\vee_{\mathfrak{S}}(\mathfrak{M})$ and $\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}$ are finite free $\mathfrak{S}^{\mathrm{ur}}$-modules, so we have $\iota_{\mathfrak{S}}\circ\iota^\vee_{\mathfrak{S}}=\mathfrak{t}^r\mathrm{Id}$. Since $\mathfrak{t}$ is a non-zero divisor in $\mathfrak{S}^{\mathrm{ur}}$, we conclude that $\iota_{\mathfrak{S}}$ is an injection and $\mathfrak{t}^r(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^\vee_{\mathfrak{S}}(\mathfrak{M}))\subset\iota_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M})$. $\square$

Using the same idea as above, we have a similar result for $\hat{\mathfrak{M}}$.

Proposition 3.1.3. (1) *$\hat{T}(\hat{\mathfrak{M}})$ induces a natural $W(R)$-linear, G-compatible morphism*

$$\hat{\iota}:\ W(R)\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{M}}\longrightarrow W(R)\otimes_{\mathbb{Z}_p}\hat{T}^\vee(\hat{\mathfrak{M}}),\tag{3.1.1}$$

where $\hat{\mathfrak{M}}=\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$.

(2) *$\hat{\iota}\simeq W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}}$. That is, the following diagram is commutative:*

$$\begin{tikzcd}
W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M} \arrow[r, "\hat{\iota}"] & W(R)\otimes_{\mathbb{Z}_p}\hat{T}^\vee(\hat{\mathfrak{M}}) \\
W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}) \arrow[u, "\alpha\otimes\mathrm{Id}_{\mathfrak{M}}", "\wr"'] \arrow[r, "W(R)\otimes_\varphi\iota_{\mathfrak{S}}"] & W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^\vee_{\mathfrak{S}}(\mathfrak{M})) \arrow[u, "\alpha\otimes(\theta^\vee)^{-1}", "\wr"']
\end{tikzcd}$$

where two vertical arrows are isomorphisms, $\alpha:W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}\mathfrak{S}^{\mathrm{ur}}\to W(R)$ is the isomorphism given by $\alpha(\sum_i a_i\otimes b_i)=\sum_i a_i\varphi(b_i)$ with $a_i\in W(R)$, $b_i\in\mathfrak{S}^{\mathrm{ur}}$ and $W(R)\otimes_\varphi\iota_{\mathfrak{S}}$ denotes $W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}}$.

(3) *$\hat{\iota}$ is an injection and $(\varphi(\mathfrak{t}))^r(W(R)\otimes_{\mathbb{Z}_p}\hat{T}^\vee(\hat{\mathfrak{M}}))\subset\hat{\iota}(W(R)\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{M}})$.*

[6]Here we use slightly different notations from those in [Liu07].

Proof. (1) We use the same idea for the construction of $\iota_{\mathfrak{S}}$ in Proposition 3.2.1 in [Liu07]. One first proves that

$$\hat{T}(\hat{\mathfrak{M}}) \simeq \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}, W(R))$$

is an isomorphism of G-modules, where the G-action on the right side is given by $g(f)(\cdot) = g(f(g^{-1}(\cdot)))$, for any $g \in G$ and $f \in \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}, W(R))$. Then we have a map

$$\hat{\iota} : \ W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \to \mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}(\hat{\mathfrak{M}}), W(R)) = W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\hat{\mathfrak{M}})$$

induced by $x \mapsto (f \mapsto f(x), \ \forall f \in \hat{T}(\hat{\mathfrak{M}}))$ for any $x \in \hat{\mathfrak{M}}$. It is easy to check that $\hat{\iota}$ is compatible with G-actions on both sides.

(2) By identifying $W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\hat{\mathfrak{M}})$ with $\mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}(\hat{\mathfrak{M}}), W(R))$ and identifying $W(R) \otimes_{\varphi, \mathfrak{S}^{\mathrm{ur}}} (\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M}))$ with $\mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}(\mathfrak{M}), W(R) \otimes_{\varphi, \mathfrak{S}^{\mathrm{ur}}} \mathfrak{S}^{\mathrm{ur}})$, it suffices to prove the following diagram is commutative:

$$\begin{tikzcd}
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r, "\hat{\iota}"] & \mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}(\hat{\mathfrak{M}}), W(R)) \\
W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} (\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathfrak{S}} \mathfrak{M}) \arrow[u, "\alpha\otimes \mathrm{Id}_{\mathfrak{M}}", "\wr"'] \arrow[r, "W(R)\otimes_\varphi \iota_{\mathfrak{S}}"] & \mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}(\mathfrak{M}), W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \mathfrak{S}^{\mathrm{ur}}) \arrow[u, "(\theta,\alpha)", "\wr"']
\end{tikzcd}$$

where the right vertical arrow (θ, α) is the isomorphism induced by θ^{-1} and α. Identify $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ with $W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} (\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathfrak{S}} \mathfrak{M})$ via $\alpha \otimes \mathrm{Id}_{\mathfrak{M}}$. For any $y = \sum_i a_i \otimes m_i \in W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ with $a_i \in W(R)$ and $m_i \in \mathfrak{M}$, we see that $(W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}})(y)$ sends $f \in T_{\mathfrak{S}}(\mathfrak{M})$ to $\sum_i a_i \otimes f(m_i)$. Then we have that $(\theta,\alpha)((W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}})(y))$ sends $\theta(f)$ to $\sum_i a_i \varphi(f(m_i))$. On the other hand, we have $\hat{\iota}(y)$ sends $\theta(f)$ to $\sum_i a_i \theta(f)(m_i) = \sum_i a_i \varphi(f(m_i))$. Hence $W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}} \simeq \hat{\iota}$.

(3) Select an $\mathfrak{S}$-basis $e_1, \dots, e_d$ of $\mathfrak{M}$ and a $\mathbb{Z}_p$-basis $f_1, \dots, f_d$ of $T^\vee_{\mathfrak{S}}(\mathfrak{M})$. We have $\iota_{\mathfrak{S}}(e_1, \dots, e_d) = (f_1, \dots, f_d)A$ with A a $d \times d$-matrix and coefficients of A in $\mathfrak{S}^{\mathrm{ur}}$. Since $\iota_{\mathfrak{S}}$ is injective and $\mathfrak{t}^r(\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M})) \subset \iota_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathfrak{S}} \mathfrak{M})$ by Lemma 3.1.2, there exists a matrix B with coefficients in $\mathfrak{S}^{\mathrm{ur}}$ such that $AB = \mathfrak{t}^r I$. Since $\hat{\iota} \simeq W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}$ by (2), by identifying $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and identifying $T^\vee_{\mathfrak{S}}(\mathfrak{M})$ with $\hat{T}^\vee(\hat{\mathfrak{M}})$ via $(\theta^\vee)^{-1}$, we have $\hat{\iota}(e_1, \dots, e_d) = (f_1, \dots, f_d)\varphi(A)$. Since $\varphi(A)\varphi(B) = (\varphi(\mathfrak{t}))^r I$ and $\varphi(\mathfrak{t})$ is a nonzero divisor in $W(R)$, we see that $\hat{\iota}$ is an injection and $(\varphi(\mathfrak{t}))^r (W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\hat{\mathfrak{M}})) \subset \hat{\iota}(W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}})$. $\square$

Remark 3.1.4. Let V be a representation of $E(u)$-height r, T a G-stable $\mathbb{Z}_p$-lattice in V, and $\mathfrak{M}$ the Kisin module associated to $T|_{G_\infty}$. We can always consider the injection

$$\tilde{\iota} := W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}} : \ W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \hookrightarrow W(R) \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M}).$$

There is a natural G-action on the right side because T is G-stable. In general, it is not clear whether the image of $\tilde{\iota}$ is G-stable under this action (though it is G_∞-stable), or equivalently, whether the G-orbit of $\mathfrak{M}$, $G(\mathfrak{M}) \subset W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. As we will see soon, in the case of $(\varphi, \hat{G})$-modules we have $G(\mathfrak{M}) \subset \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \subset W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. This is actually a key point to prove that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ is semi-stable.

Now we are ready to prove that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ is semi-stable. Tensoring B^+_{cris} on both sides of (3.1.1), noting that

$$B^+_{\mathrm{cris}} \otimes_{W(R)} W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} = B^+_{\mathrm{cris}} \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} = B^+_{\mathrm{cris}} \otimes_{\mathcal{R}_{K_0}} \mathcal{R}_{K_0} \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}$$

and $\mathcal{R}_{K_0} \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \simeq \mathcal{R}_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, we have

$$B^+_{\mathrm{cris}} \otimes_{W(R)} \hat{\iota} : \ B^+_{\mathrm{cris}} \otimes_{\mathcal{R}_{K_0}} (\mathcal{R}_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \to B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\hat{\mathfrak{M}}). \tag{3.1.2}$$

By a similar argument for $\iota_{\mathfrak{S}}$, we also have

$$B^+_{\mathrm{cris}} \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}} : \ B^+_{\mathrm{cris}} \otimes_{\mathcal{R}_{K_0}} (\mathcal{R}_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \to B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M}). \tag{3.1.3}$$

Since $\hat{\iota} \simeq W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}$ by Proposition 3.1.3, we have the following commutative diagram to identify (3.1.2) with (3.1.3):

$$\begin{tikzcd}
B^+_{\mathrm{cris}} \otimes_{\mathcal{R}_{K_0}} (\mathcal{R}_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \arrow[r, "(3.1.2)"] \arrow[d, "\mathrm{Id}"', "\wr"] & B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\hat{\mathfrak{M}}) \arrow[d, "\wr"', "B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \theta^\vee"] \\
B^+_{\mathrm{cris}} \otimes_{\mathcal{R}_{K_0}} (\mathcal{R}_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \arrow[r, "(3.1.3)"] & B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M}).
\end{tikzcd}$$

Thus when we equip $T^\vee_{\mathfrak{S}}(\mathfrak{M})$ with G-action induced by $\theta^\vee$, we see that $\mathcal{R}_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ in (3.1.3) is G-stable and has the same $\hat{G}$-action as that on $\mathcal{R}_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ in (3.1.2). Now the proof of semi-stability of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ will be almost the same as in §7 of [Liu07] *carefully using the hypothesis in Definition 2.2.3 (5): $\hat{G}$ acts on the $W(k)$-module $M := \hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}}$ trivially.* In fact, the proof in §7 of [Liu07] has a gap without verifying this hypothesis. Here we sketch the proof in §7 of [Liu07] here and indicate where the gap is. While §7 in [Liu07] only deal with the case $p > 2$, we will also discuss the case $p = 2$ here (and in §4.2).

In §7 in [Liu07], we also aim to prove that a certain representation V of $E(u)$-height r is semi-stable with Hodge-Tate weights in $\{0, \dots, r\}$ (for $p > 2$). Other than requiring that V is of finite $E(u)$-height such that we can establish (3.1.3), the only other inputs that §7 needs are three conditions listed in the beginning of §7.1 on the $\hat{G}$-action on $\mathcal{D} \otimes_S \mathcal{R}_{K_0} \simeq \mathcal{R}_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, where $\mathcal{D} := S_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, together with an extra condition that will be explained below. But the first three conditions are just conditions (2), (3), (4) required in Definition 2.2.3. Thus the same proof will go through once we have explained how the extra condition is implied by Definition 2.2.3 (5), as well as how the proof must be modified when $p = 2$.

More precisely, let $D := \mathcal{D}/(I_+S_{K_0})\mathcal{D}$ (recall $S_{K_0} = S[\frac{1}{p}]$). Then D is a finite free K_0-module with a semi-linear Frobenius action φ. One can prove there is a unique φ-equivariant section $D \hookrightarrow \mathcal{D}$ (cf. Lemma 7.3.1 in [Liu07]). So we can regard D as a K_0-submodule in $\mathcal{D}$. Since $D \hookrightarrow \mathcal{D}$ is φ-equivariant, Lemma 7.1.3 in [Liu07] showed that the structure of $\mathcal{R}_{K_0}$ forces that $\hat{G}(D) \subset (K_0[\![t]\!] \cap \mathcal{R}_{K_0}) \otimes_{K_0} D$ ($\mathcal{R}_{K_0}$ can be regarded as a subring of $K_0[\![t, u]\!]$ via Lemma 7.1.2 in [Liu07]). Note that though Lemma 7.1.3 only proved that $\tau(D) \subset (K_0[\![t]\!] \cap \mathcal{R}_{K_0}) \otimes_{K_0} D$ for a (specially selected) $\tau \in G_{p^\infty}$, the proof actually works for any $g \in \hat{G}$. In particular, we see that $\hat{G}(D) \subset (K_0[\![t]\!] \cap \mathcal{R}_{K_0}) \otimes_{K_0} D$ if $p = 2$.

To show that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ is semi-stable, at this point the proofs for case $p > 2$ and case $p = 2$ become different, we address the case $p > 2$ now and deal with the case $p = 2$ in §4.2.

Suppose $p>2$. By Lemma 5.1.2 in [Liu08b], we have $K_{p^\infty}\cap K_\infty = K$, $\hat{G} = G_{p^\infty}\rtimes H_K$ and $G_{p^\infty}\simeq \mathbb{Z}_p(1)$. As in the proof of Proposition 7.1.1 in [Liu07], let τ be a topological generator of G_{p^∞} and select a basis $e_1,\dots,e_d$ of D and write

$$\tau(e_1,\dots,e_d)=(e_1,\dots,e_d)A,\quad A=A(t)=\sum_{i=0}^{\infty}A_i\gamma_i(t).$$

Now the facts that $\hat{G}(D)\subset (K_0[\![t]\!]\cap\mathcal{R}_{K_0})\otimes_{K_0}D$ and H_K acts on D trivially imply (cf. the proof of Proposition 7.1.1 in [Liu07])

$$A(\chi(g)t)=A(t)^{\chi(g)}, \tag{3.1.4}$$

where χ is the p-adic cyclotomic character. Since $\hat{G}$ acts on $\hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}}$ trivially, $A(0)=\mathrm{Id}$. Thus $\log(A(t))$ is well defined and $\log(A(\chi(g)t))=\chi(g)\log(A(t))$. Selecting a $g\in\hat{G}$ such that $\chi(g)\neq 1$, we see that there exists a matrix N (hence a linear map $N: D\to D$) such that $\log(A(t))=Nt$, thus $\tau(x)=\sum\limits_{n=0}^{\infty}\gamma_i(t)\otimes N^i(x)$ for any $x\in D$. Now consider the K_0-vector space

$$\bar{D}:=\left\{\sum_{i=0}^{\infty}\gamma_i(\mathfrak{u})\otimes N^i(x)\in B^+_{\mathrm{st}}\otimes_S\mathcal{D}|x\in D\right\}$$

where $\mathfrak{u}=\log(u)\in B^+_{\mathrm{st}}$. Note that $\tau(\mathfrak{u})=\mathfrak{u}+\log(\underline{\epsilon}(\tau))$. Since $\hat{G}\simeq G_{p^\infty}\rtimes H_K$ and τ is a topological generator of G_{p^∞}, we see that $\underline{\epsilon}(\tau)=(\epsilon_i(\tau))_{i\geq 0}\in R$ with $\epsilon_i(\tau)$ a primitive p^i-th root of unity. Hence we can select $t=-\log([\underline{\epsilon}(\tau)])$ and then $\tau(\mathfrak{u})=\mathfrak{u}-t$. As in §7.2 in [Liu07], we can show that $\bar{D}\subset (B^+_{\mathrm{st}}\otimes_S\mathcal{D})^G\subset (B^+_{\mathrm{st}}\otimes_{\mathbb{Z}_p}\hat{T}^\vee(\hat{\mathfrak{M}}))^G$. But $\dim_{K_0}\bar{D}=\dim_{K_0}D=\mathrm{Rank}_{\mathbb{Z}_p}\hat{T}(\hat{\mathfrak{M}})$. Therefore $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\hat{T}(\hat{\mathfrak{M}})$ is semi-stable and the functor $\hat{T}$ is well-defined.

Remark 3.1.5. If we do not assume that $\hat{G}$ acts on $\hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}}$ trivially, or equivalently $A(0)=\mathrm{Id}$, then $\log(A(t))$ may not be well-defined and $A(t)$ may not be $\sum\limits_{n=0}^{\infty}N^i\gamma_i(t)$. Consequently, $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\hat{T}(\hat{\mathfrak{M}})$ may not be semi-stable; see Example 4.2.1. So this is the gap in §7.1 of [Liu07], on the 3rd to last line of page 668 of the published version. We will close this gap in the end of this note.

Now let us prove the full faithfulness of $\hat{T}$ (including the case $p=2$). Suppose that $f: T'\to T$ is a morphism of G-stable $\mathbb{Z}_p$-lattices inside semi-stable representations, and there exist $(\varphi,\hat{G})$-modules $\hat{\mathfrak{M}}'=(\mathfrak{M}',\varphi_{\mathfrak{M}'},\hat{G}_{\mathfrak{M}'})$ and $\hat{\mathfrak{M}}=(\mathfrak{M},\varphi_{\mathfrak{M}},\hat{G}_{\mathfrak{M}})$ such that $\hat{T}(\hat{\mathfrak{M}}')\simeq T'$ and $\hat{T}(\hat{\mathfrak{M}})\simeq T$. Note that $T_{\mathfrak{S}}$ is fully faithful (Theorem 2.1.1), so there exists a morphism of Kisin modules $\mathfrak{f}:\mathfrak{M}\to\mathfrak{M}'$ such that $T_{\mathfrak{S}}(\mathfrak{f})=f|_{G_\infty}$. So we obtain the following commutative diagram

$$\begin{tikzcd}
\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M} \arrow[r, "\iota_{\mathfrak{S}}"] \arrow[d, "\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{f}"] & \mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^\vee_{\mathfrak{S}}(\mathfrak{M}) \arrow[d, "\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}f^\vee"] \\
\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}' \arrow[r, "\iota'_{\mathfrak{S}}"] & \mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^\vee_{\mathfrak{S}}(\mathfrak{M}').
\end{tikzcd}$$

By Lemma 3.1.1, it suffices to show that $\hat{\mathfrak{f}}:=\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{f}$ is G-equivariant. To see this, tensoring $W(R)$ via $\mathfrak{S}^{\mathrm{ur}}\overset{\varphi}{\hookrightarrow}W(R)$ to the commutative diagram and noting that

$\hat{\iota} \simeq W(R) \otimes_{\varphi, \mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}$ by Proposition 3.1.3 (2), we have the following commutative diagram

$$\begin{tikzcd} W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \arrow[r, "\hat{\iota}"] \arrow[d, "W(R)\otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{f}}"] & W(R) \otimes_{\mathbb{Z}_p} \hat{T}^{\vee}(\hat{\mathfrak{M}}) \arrow[d, "W(R)\otimes_{\mathbb{Z}_p} f^{\vee}"] \\ W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}' \arrow[r, "\hat{\iota}'"] & W(R) \otimes_{\mathbb{Z}_p} \hat{T}^{\vee}(\hat{\mathfrak{M}}'). \end{tikzcd}$$

Note that $\hat{\iota}$ is injective and G-equivariant by Proposition 3.1.3. Since f is G-equivariant, $\hat{\mathfrak{f}}$ is G-equivariant.

3.2. **The essential surjectiveness of $\hat{T}$.** Now assume that T is a G-stable $\mathbb{Z}_p$-lattice in a semi-stable representation V with Hodge-Tate weights in $\{0, \dots, r\}$. By Theorem 2.1.1, there exists a Kisin module $\mathfrak{M}$ such that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T|_{G_\infty}$. Theorem 5.4.2 in [Liu07] showed that the injection

$$B^+_{\mathrm{cris}} \otimes_{\varphi, \mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}} : \ B^+_{\mathrm{cris}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \longrightarrow B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T^{\vee}_{\mathfrak{S}}(\mathfrak{M}) = B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T^{\vee}$$

is compatible with the G-actions. More precisely, let $\mathcal{D} := S_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ be the Breuil module[7] associated to V, and let N be the monodromy operator on $\mathcal{D}$. Then G acts on $B^+_{\mathrm{cris}} \otimes_{S_{K_0}} \mathcal{D} = B^+_{\mathrm{cris}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ via (cf. equation (5.2.1) in [Liu07])

$$g(a \otimes x) = \sum_{i=0}^{\infty} g(a) \gamma_i(-\log([\underline{\epsilon}(g)])) \otimes N^i(x). \tag{3.2.1}$$

for any $a \in B^+_{\mathrm{cris}}$ and $x \in \mathcal{D}$.

Set $\varphi^* \mathfrak{M} = \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. Apparently $\varphi^* \mathfrak{M}$ is a finite free $\mathfrak{S}$-module and $\varphi^* \mathfrak{M} \subset \mathcal{D} \subset B^+_{\mathrm{cris}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. Now we are interested in the orbit $G(\varphi^* \mathfrak{M})$ of $\varphi^* \mathfrak{M}$ under G.

Proposition 3.2.1. $G(\varphi^* \mathfrak{M}) \subset \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$.

Proof. We have seen that $G(\varphi^* \mathfrak{M}) \subset G(\mathcal{D}) \subset \mathcal{R}_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ by formula (3.2.1). Now consider the following commutative diagram

$$\begin{tikzcd} \varphi^* \mathfrak{M} \arrow[r, hook] \arrow[d, equal] & W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \arrow[r] \arrow[d, hook] & W(R) \otimes_{\mathbb{Z}_p} T^{\vee} \arrow[d, hook] \\ \varphi^* \mathfrak{M} \arrow[r, hook] & B^+_{\mathrm{cris}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \arrow[r] & B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T^{\vee} \end{tikzcd} \tag{3.2.2}$$

where the first row is obtained by $W(R) \otimes_{\varphi, \mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}$. Obviously, the right column is compatible with the G-actions. By Lemma 3.1.2, for any $g \in G$, we have

$$(\varphi(\mathfrak{t}))^r (g(\varphi^* \mathfrak{M})) \subset W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}.$$

Now select a basis $e_1, \dots, e_d$ of $\varphi^* \mathfrak{M}$ and write $g(e_1, \dots, e_d) = (e_1, \dots, e_d) A$ with A a $d \times d$-matrix. Let a be an entry of A. It suffices to show that $a \in W(R)$. Now we know that $a \in \mathcal{R}_{K_0}$ and $(\varphi(\mathfrak{t}))^r a \in W(R)$. Then we can reduce the proof to Lemma 3.2.2 below.

□

[7]A Breuil module is a finite free S_{K_0}-module with structures of Frobenius, filtration and monodromy. By [Bre97], the category of admissible Breuil modules is equivalent to the category of semi-stable representations. Also see §3.2 in [Liu08b] for the relation between Kisin modules and Breuil modules.

Lemma 3.2.2. *Let $a \in B^+_{\mathrm{cris}}$. If $(\varphi(\mathfrak{t}))^r a \in W(R)$ then $a \in W(R)$.*

Proof. Without loss of generality, we may assume that $r = 1$. Recall the map $\theta : A_{\mathrm{cris}} \to \widehat{\mathcal{O}}_{\overline{K}}$ constructed in the beginning of §2.2. Define an ideal $\mathrm{Fil}^1 B^+_{\mathrm{cris}} := \mathrm{Ker}(\theta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \subset B^+_{\mathrm{cris}}$ and set $\mathrm{Fil}^1 W(R) = \mathrm{Fil}^1 B^+_{\mathrm{cris}} \cap W(R)$. As in [Fon94a], §5.1, define

$$I^{[1]}W(R) = \{a \in W(R) | \varphi^n(a) \in \mathrm{Fil}^1 W(R), \text{ for every } n \geq 0\}.$$

Write $x = \varphi(\mathfrak{t})a$. We claim that $x \in I^{[1]}W(R)$. By Example 5.3.3 in [Liu07] or Example 3.2.3 below, there exists a unit $c \in A_{\mathrm{cris}}$ such that $t = c\varphi(\mathfrak{t})$. Since $\theta([\underline{\epsilon}] - 1) = 0$, we see that $t \in \mathrm{Fil}^1 B^+_{\mathrm{cris}}$ and then $x = \varphi(\mathfrak{t})a \in \mathrm{Fil}^1 W(R)$. Since $\varphi(\mathfrak{t}) = c_0^{-1}E(u)\mathfrak{t}$, we have $\varphi^n(\mathfrak{t}) = (\prod\limits_{i=1}^{n-1} \varphi^i(c_0^{-1}E(u)))\varphi(\mathfrak{t}) \in \mathrm{Fil}^1 W(R)$. Therefore $\varphi^n(x), \varphi^n(\varphi(\mathfrak{t})) \in \mathrm{Fil}^1 W(R)$ and then $x, \varphi(\mathfrak{t}) \in I^{[1]}W(R)$. By proposition 5.1.3 in [Fon94a], $I^{[1]}W(R)$ is a principal ideal and $b \in I^{[1]}W(R)$ is a generator if and only if $v_R(\tilde{b}) = \frac{p}{p-1}$, where $\tilde{b} = b \mod p \in R$. Write $\tilde{\mathfrak{t}} = \mathfrak{t} \mod p \in R$. Since $\varphi(\mathfrak{t}) = c_0^{-1}E(u)\mathfrak{t}$ and $\mathfrak{t} \notin p\mathfrak{S}^{\mathrm{ur}}$, we easily compute that $v_R(\tilde{\mathfrak{t}}) = \frac{1}{p-1}$ and then $v_R(\widetilde{\varphi(\mathfrak{t})}) = \frac{p}{p-1}$. Thus $\varphi(\mathfrak{t})$ is a generator of $I^{[1]}W(R)$. So $a = x/\varphi(\mathfrak{t}) \in W(R)$. $\square$

Now Proposition 3.2.1 implies that $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is stable under the G-action induced from $B^+_{\mathrm{cris}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \hookrightarrow B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T^\vee$, and obviously the G-action on $\hat{\mathfrak{M}}$ factors through $\hat{G}$. It suffices to check that $\hat{G}$ acts on $\hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}}$ trivially to show that $\hat{\mathfrak{M}}$ is a $(\varphi, \hat{G})$-module. Note that $[\underline{\epsilon}(g)] - 1 \in I_+W(R)$ and then $\log([\underline{\epsilon}(g)]) \in I_+A_{\mathrm{cris}}$. Thus by formula (2.2.1), we easily check that $\hat{G}$ acts on $\mathcal{R}_{K_0}/I_+\mathcal{R}_{K_0}$ trivially. By formula (3.2.1), we see that $\hat{G}$ acts on $\mathcal{D}/(I_+S_{K_0})\mathcal{D} = \hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ trivially. Hence $\hat{\mathfrak{M}}$ is a $(\varphi, \hat{G})$-module.

Now it remains to check that $\hat{T}(\hat{\mathfrak{M}}) \simeq T$. First, by Lemma 3.1.1, $\hat{T}(\hat{\mathfrak{M}})|_{G_\infty} \simeq T|_{G_\infty}$. Recall that $\hat{\iota}$ defined in (3.1.1) is compatible with the G-actions on both sides, and $\hat{\iota} \simeq W(R) \otimes_{\varphi,\mathfrak{S}} \iota_{\mathfrak{S}}$. Comparing $\hat{\iota}$ with the top row of (3.2.2), we have the following commutative diagram:

$$\begin{tikzcd}
W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \arrow[r, "\hat{\iota}"] \arrow[d, "\mathrm{Id}"', "\wr"] & W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\hat{\mathfrak{M}}) \arrow[d, "\wr"', "W(R)\otimes_{\mathbb{Z}_p}\theta^\vee"] & \\
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r, "W(R)\otimes_\varphi \iota_{\mathfrak{S}}"] & W(R) \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M}) \arrow[r, equal] & W(R) \otimes_{\mathbb{Z}_p} T^\vee,
\end{tikzcd}$$

where $W(R) \otimes_\varphi \iota_{\mathfrak{S}}$ denotes $W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}$. By the construction of $\hat{\mathfrak{M}}$, we see that the left column is compatible with the G-actions. By Proposition 3.1.3, $(\varphi(\mathfrak{t}))^r(W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})) \subset \hat{\iota}(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$. So the right column is also compatible with the G-actions. Therefore, $\hat{T}(\hat{\mathfrak{M}}) \simeq T$ as G-modules. This completes the proof of the main theorem.

Unlike $\mathcal{R}_{K_0}$, so far we do not have an explicit description of $\widehat{\mathcal{R}}$. Recall $\underline{\epsilon} = (\zeta_{p^i})_{i \geq 0} \in R$ with ζ_{p^i} a primitive p^i-th root of unity. Put

$$w := [\underline{\epsilon}] - 1 = \exp(t) - 1.$$

We see that $w \in \widehat{\mathcal{R}}$ and $W(k)[\![u, w]\!] \subset \widehat{\mathcal{R}}$ is stable under Frobenius and $\hat{G}$-action. Unfortunately, this inclusion is strict. The following example shows that the structure of $\widehat{\mathcal{R}}$ may be very complicated.

Example 3.2.3. Assume that $p > 2$ and let τ be a topological generator of G_{p^∞}. It is well known that t is the period of the cyclotomic character χ. On the other hand, $\mathfrak{t}$ is the period of the Kisin module for χ, which is an $\mathfrak{S}$-free rank-1 module $\mathfrak{S}^* := \mathfrak{S} \cdot f$ and $\varphi(f) = c_0^{-1} E(u) f$ with f a basis. Example 5.3.3 in [Liu07] showed that we may choose t such that $t = c\varphi(\mathfrak{t})$, where $c = \prod\limits_{i=0}^{\infty} \varphi^n(\frac{\varphi(c_0^{-1}E(u)}{p}) \in A^*_{\mathrm{cris}}$. Then $\tau(c)\tau(\varphi(\mathfrak{t})) = \tau(t) = t = c\varphi(\mathfrak{t})$. Therefore $\tau(\varphi(\mathfrak{t})) = \frac{c}{\tau(c)}\varphi(\mathfrak{t}) = \prod\limits_{n=1}^{\infty} \varphi^n(\frac{E(u)}{\tau(E(u))})\varphi(\mathfrak{t})$. Let $\hat{c} = \frac{c}{\tau(c)} = \prod\limits_{n=1}^{\infty} \varphi^n(\frac{E(u)}{\tau(E(u))})$. Since c is a unit in $\mathcal{R}_{K_0}$, $\hat{c} \in \mathcal{R}_{K_0}$. On the other hand, $E(u)$ is a generator of $\mathrm{Fil}^1 W(R)$, so $\frac{E(u)}{\tau(E(u))}$ is a unit in $W(R)$. Thus $\hat{c} \in \widehat{\mathcal{R}}$. Let $\widehat{\mathfrak{S}}^* := (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{S}) \cdot f = \widehat{\mathcal{R}} \cdot f$ be the $(\varphi, \hat{G})$-module corresponding to χ. Then the $\hat{G}$-action on $\widehat{\mathfrak{S}}^*$ is given by $\tau(f) = \hat{c} f$.

4. Weak $(\varphi, \hat{G})$-modules

A triple $(\mathfrak{M}, \varphi, \hat{G})$ is called a *weak* $(\varphi, \hat{G})$*-module* if it only satisfies axioms (1), (2), (3) and (4) in Definition 2.2.3. We define morphisms of weak $(\varphi, \hat{G})$-modules and the functor $\hat{T}$ to be the same as those of $(\varphi, \hat{G})$-modules. In this section, we will discuss the relation between weak $(\varphi, \hat{G})$-modules and potentially semi-stable representations and complete the proof of Theorem 2.3.1 (for $p = 2$). First we need some preparations for the case $p = 2$.

4.1. **The structure of $\hat{G}$ when $p = 2$.** Recall that Lemma 5.1.2 in [Liu08b] showed that if $p > 2$ then $K_{p^\infty} \cap K_\infty = K$ and consequently, $\hat{G} = G_{p^\infty} \rtimes H_K$. This statement fails in general when $p = 2$. Set $K_n = K(\pi_n)$ for $n \geq 1$. We assume $p = 2$ for the remainder of this subsection.

Example 4.1.1. Let $F_n := \mathbb{Q}_2(\alpha_n)$ with $\alpha_n = \zeta_{2^n} + \zeta_{2^n}^{-1}$ and $n \geq 3$. We have $\alpha_{n+1}^2 = \alpha_n + 2$. So it is easy to check that α_n is a uniformizer of F_n. Assume that $K = F_n$ and select $\pi = \alpha_n + 2$. We see that $K_1 = K(\sqrt{\pi}) = F_{n+1} \subset \mathbb{Q}_2(\zeta_{2^{n+1}})$.

Remark 4.1.2. From the above example, we see that Lemma 8.0.4 in [Liu07] is false. Hence the whole proof in §8 in [Liu07] has to be fixed. We will fix the problem in the end of this note (Errata for [Liu07]).

Lemma 4.1.3. *Let F be a nontrivial K-subfield of K_n, that is, $K \subset F \subset K_n$ and $F \neq K$. Then $F = K_i$ with $i \leq n$.*

Proof. Let us write $K_0 = K$ only in the proof of this lemma and prove the lemma by induction on n. It is trivial if $n = 1$. Assume that the statement is valid for $n = m$. Let us consider the K-subfield F of K_{m+1}. If $F \neq K_{m+1}$ and $F \not\subset K_m$, then $FK_m = K_{m+1}$. Since $K_m \cap F$ is a subfield of K_m, by induction, $K_m \cap F = K_l$ for $l < m$. We claim there exists a quadratic extension F_2 over K_l such that $F_2 \subset F$. To see this, let L be the Galois closure of K_{m+1}. Note that $F \subset L$ and $\mathrm{Gal}(L/K)$ has an order of power 2. Then the claim follows the fact that any proper subgroup H of a group G with the order of power p admits a normal subgroup

H' of G such that $H \subset H'$ and $[G : H'] = p$ (see Theorem 5.7 in Chapter II in [Hun80]). Since F_2 is quadratic and $F_2 \cap K_m = K_l$, we easily see that $K_{m+1} = F_2 K_m$ and $\mathrm{Gal}(K_{m+1}/K_m) \simeq \mathrm{Gal}(F_2/K_l)$. Then $F_2 = K_l(\sqrt{a})$ with $a \in K_l$ and $K_{m+1} = K_m F_2 = K_m(\sqrt{a})$. Hence $\pi_{m+1} = x\sqrt{a} + y$ with $x, y \in K_m$. Write $\sigma \in \mathrm{Gal}(K_{m+1}/K_m)$ such that $\sigma \neq \mathrm{Id}$. Note that $\mathrm{Gal}(K_{m+1}/K_m) \simeq \mathrm{Gal}(F_2/K_l)$, we have $-\pi_{m+1} = \sigma(\pi_{m+1}) = x\sigma(\sqrt{a}) + y = -x\sqrt{a} + y$. Thus $y = 0$ and $\pi_{m+1} = x\sqrt{a}$. Normalize the valuation v_p such that $v_p(\pi) = 1$. Then

$$\frac{1}{2^{m+1}} = v_p(\pi_{m+1}) = v_p(x) + \frac{1}{2} v_p(a).$$

Since $x \in K_m$, $a \in K_l$ and $l < m$, the right-hand side is in $\frac{1}{2^m}\mathbb{Z}$. Contradiction. So F_2 can not exist and consequently F can not exist. Hence $F = K_{m+1}$ or $F \subset K_m$. So by induction, we have all nontrivial K-subfields of K_{m+1} are K_i with $i \leq m+1$. □

Remark 4.1.4. The above Lemma is still valid if $p > 2$. Since $K_{p^\infty} \cap K_\infty = K$, it suffices to show that for any field F satisfying $K(\zeta_{p^n}) \subsetneq F \subset K(\zeta_{p^n}, \pi_n)$, we have $F = K(\zeta_{p^n}, \pi_l)$ for some $l \leq n$. This follows the fact that $\mathrm{Gal}(K(\zeta_{p^n}, \pi_n)/K(\zeta_{p^n})) \simeq \mathbb{Z}/p^n\mathbb{Z}$.

Proposition 4.1.5. *Let $K' := K_{p^\infty} \cap K_\infty$. If $K' \neq K$ then $K' = K_1$ and $\zeta_4 \notin K'$.*

Proof. We first prove that if $\zeta_4 \in K$ then $K' = K$. Let $\tilde{K} = K \cap \mathbb{Q}_{2,2^\infty}$. Then $\mathrm{Gal}(K_{p^\infty}/K) \simeq \mathrm{Gal}(\mathbb{Q}_{2,2^\infty}/\tilde{K}) \subset \mathrm{Gal}(\mathbb{Q}_{2,2^\infty}/\mathbb{Q}_2(\zeta_4)) \simeq \mathbb{Z}_2$. So $\tilde{K} = \mathbb{Q}_2(\zeta_{2^m})$ with $m \geq 2$, and if F is a quadratic extension of K and $F \subset K_{p^\infty}$ then $F = K(\zeta_{2^{m+1}})$. If $K' \neq K$ then by Lemma 4.1.3, we see that $K' = K_l$ for some $l \geq 1$. Hence $K_1 \subset K_{p^\infty}$ is a quadratic extension of K. So $K_1 = K(\zeta_{2^{m+1}})$ and then $\pi_1 = x\zeta_{2^{m+1}} + y$ with $x, y \in K$. Let σ be the nontrivial element in $\mathrm{Gal}(K_1/K)$. We have $-\pi_1 = \sigma(\pi_1) = x\sigma(\zeta_{2^{m+1}}) + y = -x\zeta_{2^{m+1}} + y$. So $\pi_1 = x\zeta_{2^{m+1}}$ and then $v_p(\pi_1) = v_p(x)$. This is impossible because π_1 is a uniformizer of K_1. Hence this forces $K' = K$.

Now if $K' \neq K$ then $\zeta_4 \notin K$. By Lemma 4.1.3, we see that $K' = K_l$ with $l \geq 1$. If $l \geq 2$ then $K_2 \subset K_{p^\infty}$ is an abelian extension of K with degree 4. Note that $K_2 = K(\pi_2)$, so there exists at least a $g \in \mathrm{Gal}(K_2/K)$ such that $g(\pi_2) = \zeta_4\pi_2$ (otherwise, $\mathrm{Gal}(K_2/K)$ can not have order 4). Hence $\zeta_4 \in K_2$. Note that $\zeta_4 \notin K$, we have $F = K(\zeta_4) \subset K_2$ is a quadratic extension of K. Hence by Lemma 4.1.3, we have $F = K_1$. So $\pi_1 = x\zeta_4 + y$ for $x, y \in K$. Let σ be the nontrivial element in $\mathrm{Gal}(K_1/K)$. We have $-\pi_1 = \sigma(\pi_1) = x\sigma(\zeta_4) + y = -x\zeta_4 + y$. So $\pi_1 = x\zeta_4$ and then $v_p(\pi_1) = v_p(x)$. This is impossible because π_1 is a uniformizer of K_1. So this forces $l = 1$, $K' = K_1$ and $\zeta_4 \notin K'$. □

In summary, the structure of $\hat{G}$ has two possible cases if $p = 2$. Case 1: $K' = K$. In this case, we have $\hat{G} \simeq G_{p^\infty} \rtimes H_K$ with $G_{p^\infty} \simeq \mathbb{Z}_p$ and H_K acts on G_{p^∞} via the p-adic cyclotomic character χ. In particular, we can select a topological generator $\tau \in G_{p^\infty}$ and $\log([\underline{\epsilon}(\tau)]) = \alpha t$ with $\alpha \in \mathbb{Z}_p^\times$; Case 2: $K' = K_1$. In this case, write $\hat{G}_1 := \mathrm{Gal}(\hat{K}/K_1)$. We have $\hat{G}_1 \simeq G_{p^\infty} \rtimes H_K$ with $G_{p^\infty} \simeq \mathbb{Z}_p$ and H_K acts on G_{p^∞} via the p-adic cyclotomic character χ. Let τ_1 be a topological generator of G_{p^∞}, note that $\log([\underline{\epsilon}(\tau_1)]) = 2\beta t$ with $\beta \in \mathbb{Z}_p^\times$. Nevertheless there exists a $\tau \in \hat{G}$ such that τ acts on K_1 nontrivially. Then it is easy to see that $\log([\underline{\epsilon}(\tau)]) = \beta' t$ with $\beta' \in \mathbb{Z}_p^\times$, but τ can not be chosen to be in G_{p^∞} in this case.

4.2. **Weak $(\varphi, \hat{G})$-modules and potentially semi-stable representations.** Now we assume that p is an arbitrary prime again. Recall that a weak $(\varphi, \hat{G})$-module is a triple $(\mathfrak{M}, \varphi, \hat{G})$ which only satisfies axioms (1), (2), (3) and (4) in Definition 2.2.3, and we define morphisms of weak $(\varphi, \hat{G})$-modules and the functor $\hat{T}$ to be the same as those of $(\varphi, \hat{G})$-modules. The following is an example to show that there exists a weak $(\varphi, \hat{G})$-module which is not a $(\varphi, \hat{G})$-module. Here we thank Xavier Caruso for finding this example for us.

Example 4.2.1. Let $K = \mathbb{Q}_p(\zeta_p)$ with ζ_p a p-th primitive root of unity and $\pi = \zeta_p - 1$. Then $K_1 = K(\sqrt[p]{\zeta_p - 1})$ is Galois over K. Let T be the representation of G induced by the G-action on the finite free $\mathbb{Z}_p$-module $\mathcal{O}_{K_1}$. It is easy to see that the G_∞-action on T is trivial. Let $\mathfrak{M}$ be the Kisin module attached to $T|_{G_\infty}$. We easily see that the action of G is stable over $\mathfrak{M} \hookrightarrow W(R) \otimes_{\mathbb{Z}_p} T^\vee$ and the action of G on $\mathfrak{M}/u\mathfrak{M}$ is isomorphic to the dual of the G-action on $\mathcal{O}_{K_1}$. So it is non-trivial. Therefore T is not semi-stable over K. This example also answers the question raised in Remark 2.1.2 (2): in general, a representation of finite $E(u)$-height need not be semi-stable.

Note that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ in the above is potentially semi-stable. In general, we have the following result. Set $m = \mathrm{Max}\{i | K_0(\zeta_{p^i}) \subset K\}$ and recall $K_n = K(\pi_n)$ for $n \geq 1$.

Theorem 4.2.2. *$\hat{T}$ induces a contravariant fully faithful functor from the category of weak $(\varphi, \hat{G})$-modules of height r to the category of G-stable $\mathbb{Z}_p$-lattices in potentially semi-stable representations which are semi-stable over K_m and have Hodge-Tate weights in $\{0, \dots, r\}$.*

Proof. We first prove that $\hat{T}$ is well-defined. It suffices to check that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ is semi-stable over K_m with Hodge-Tate weights in $\{0, \dots, r\}$. First we show that to prove that $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ is semi-stable over K_m, it suffices to prove that V is semi-stable over K_l with $l \geq m$. In fact, if V is semi-stable over K_l then $\mathrm{Gal}(K_l(\zeta_{p^l})/K)$ acts on $\tilde{D} := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{\tilde{G}}$ with $\tilde{G} := \mathrm{Gal}(\overline{K}/K_l(\zeta_{p^l}))$. In particular, $\mathrm{Gal}(K_l(\zeta_{p^l})/K_l)$ acts on $\tilde{D}$ trivially. Now we claim that there exists a $\sigma \in \mathrm{Gal}(K_l(\zeta_{p^l})/K)$ such that $\sigma(\pi_l) = \pi_l \zeta_{p^l}$. The claim follows the fact that there exists $\tau \in \hat{G}$ such that $\underline{\epsilon}(\tau) = (\epsilon_i)_{i \geq 0} \in R$ with ϵ_i the primitive p^i-th root of unity. If $p > 2$ then this is clear because $K_{p^\infty} \cap K_\infty = K$. If $p = 2$ then the end of §4.1 shows the existence of such a τ. Now we see that $\mathrm{Gal}(K_l(\zeta_{p^l})/K(\pi_l\zeta_{p^l})) = \sigma \,\mathrm{Gal}(K_l(\zeta_{p^l})/K_l)\sigma^{-1}$ acts on $\tilde{D}$ trivially. Set $\tilde{K} := K_l \cap K(\pi_l \zeta_{p^l})$. We see that $\mathrm{Gal}(K_l(\zeta_{p^l})/\tilde{K})$ acts on $\tilde{D}$ trivially and thus V is semi-stable over $\tilde{K}$. Now we claim that $\tilde{K} \subset K_m$. By Lemma 4.1.3 and Remark 4.1.4, we see that $\tilde{K} = K_n$ for some n. Since $\sigma : K_l \xrightarrow{\sim} K(\pi_l \zeta_{p^l})$, there exists $y \in K_l$ such that $\sigma(y) = \pi_n$. Consequently, $y = \sigma^{-1}(\pi_n) = \zeta_{p^n}^{-1}\pi_n \in K_l$, and hence $\zeta_{p^n} \in K_l$. Now it suffices to show that $\zeta_{p^n} \in K$ to conclude that V is semi-stable over K_m. If $p > 2$ then $K_{p^\infty} \cap K_\infty = K$ implies that $\zeta_{p^n} \in K$. If $p = 2$ then $\zeta_{p^n} \in K' = K_{p^\infty} \cap K_\infty$. If $K' = K$ then we have the same conclusion as $p > 2$. If $K' = K_1$ then by Proposition 4.1.5, $\zeta_4 \notin K' = K_1$. So we have $n \leq 1$. But $\zeta_2 = -1$ is always in K.

Recall $\hat{K} := K_{\infty,p^\infty}$. Set $G_l = \mathrm{Gal}(\overline{K}/K_l)$, $\hat{G}_l := \mathrm{Gal}(\hat{K}/K_l)$, $K_{l,p^\infty} := \bigcup_{n=1}^{\infty} K_l(\zeta_{p^n})$ and $G_{l,p^\infty} := \mathrm{Gal}(\hat{K}/K_{l,p^\infty})$. Let us show that V is semi-stable over K_l with some l large enough. We first deal with the case $p > 2$.

If $p > 2$ then recall that Lemma 5.1.2 in [Liu08b] showed that $\hat{G} = G_{p^\infty} \rtimes H_K$, $G_{p^\infty} \simeq \mathbb{Z}_p(1)$ and H_K acts on G_{p^∞} via the p-adic cyclotomic character χ. Let τ be a topological generator of G_{p^∞}. We easily see that $\hat{G}_l = G_{l,p^\infty} \rtimes H_K$ and G_{l,p^∞} is topologically generated by τ^{p^l}. Now using the same proof and notations in §3.1 up to formula (3.1.4), we have $A(0) = A(0)^{\chi(g)}$ for any $g \in H_K$. Choose $g \in H_K$ such that $\chi(g) = 1 + p^l$ with l large enough. We see that $A(0)^{p^l} = \mathrm{Id}$. Note τ acts on t trivially, $\tau^{p^l}(e_1, \dots, e_d) = (e_1, \dots, e_d)A^{p^l}$. Since $A^{p^l}(0) = \mathrm{Id}$, $\log(A^{p^l}(t))$ makes sense. Using the same proof below (3.1.4), we conclude that there exists a K_0-linear map $N : D \to D$ such that $\tau^{p^l}(x) = \sum_{i=0}^{\infty} \gamma_i(t) \otimes N^i(x)$ for any $x \in D$. Note that φ acts on D and $\tau\varphi = \varphi\tau$. We see that $p\varphi N = N\varphi$. Since φ is an injection, it is not hard to see that eigenvalues of N are all zeros. Thus N is nilpotent.

Now define

$$\bar{D} := \left\{ \sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}/p^l) \otimes N^i(x) \in B^+_{\mathrm{st}} \otimes_S \mathcal{D} | x \in D \right\} \tag{4.2.1}$$

where $\mathfrak{u} = \log(u) \in B^+_{\mathrm{st}}$. Since N is nilpotent, $\bar{D}$ is well-defined and we easily see $\dim_{K_0} \bar{D} = \dim_{K_0} D = \mathrm{rank}_{\mathbb{Z}_p}(\hat{T}(\hat{\mathfrak{M}}))$. It suffices to show $g(y) = y$ for any $y \in \bar{D}$ and $g \in G_l$. Since G_∞ acts on $\bar{D}$ trivially, it suffices to show that τ^{p^l} acts on $\bar{D}$ trivially. Note that $\tau^{p^l}\mathfrak{u} = \mathfrak{u} + \log([\underline{\epsilon}(\tau^{p^l})]) = \mathfrak{u} - p^l t$, we have

$$\begin{aligned}
\tau^{p^l}(y) &= \tau^{p^l}\left(\sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}/p^l) \otimes N^i(x) \right) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \gamma_i(\mathfrak{u}/p^l - t) \left(\gamma_j(t) \otimes N^{i+j}(x) \right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{i+j=k} \gamma_i(\mathfrak{u}/p^l - t)\gamma_j(t) \right) \otimes N^k(x) \\
&= \sum_{k=0}^{\infty} \gamma_k(\mathfrak{u}/p^l) \otimes N^k(x) = y.
\end{aligned}$$

Hence $\bar{D} \subset (B^+_{\mathrm{st}} \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\mathfrak{M}))^{G_l}$, and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ is semi-stable over K_l. Thus $\hat{T}$ is well-defined.

Now let us consider the case $p = 2$. Let $K' = K_{p^\infty} \cap K_\infty$. If $K' = K$ then $\hat{G} = G_{p^\infty} \rtimes H_K$ and $G_{p^\infty} \simeq \mathbb{Z}_p$. The proof in this case is the totally the same as the case $p > 2$. By Proposition 4.1.5, we only need to deal with the case that $K' = K_1 = K(\sqrt{\pi})$. Let τ_1 be a topological generator of G_{1,p^∞}. For any $l \geq 1$, we claim that $\hat{G}_l \simeq G_{l,p^\infty} \rtimes H_K$ and G_{l,p^∞} is topologically generated by $\tau_1^{p^{l-1}}$. In fact, since $K_\infty \cap K_{p^\infty} = K_1$, it is not hard to see that $K_{l,p^\infty} \cap K_\infty = K_l$ and then the claim follows. Now as in the paragraph before (3.1.4), select a basis $e_1, \dots, e_d$ of D and write $\tau_1(e_1, \dots, e_d) = (e_1, \dots, e_d)A(t)$. We still have formula (3.1.4) and then

$A(0) = A(0)^{\chi(g)}$ for any $g \in H_K$. Select a $g \in H_K$ such that $\chi(g) = 1 + 2^l$ with an $l \geq 1$. Then we see that $A^{2^l}(0) = \mathrm{Id}$. Now we can use almost the same proof as the case $p > 2$ to prove $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ is semi-stable over K_{l+1}, the only difference is that one needs to define $\bar{D} = \left\{ \sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}/2^{l+1}) \otimes N^i(x) \in B^+_{\mathrm{st}} \otimes_S \mathcal{D} | x \in D \right\}$ because $\tau_1^{2^l}(\mathfrak{u}) = \mathfrak{u} + \log([\underline{\epsilon}(\tau_1^{2^l})]) = \mathfrak{u} - 2^{l+1}t$ (we can always select t such that $2t = -\log([\underline{\epsilon}(\tau_1)])$). This finishes the proof that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ is semi-stable over K_m.

The proof of full faithfullness is exactly the same as the end of §3.1. Note the proofs of Lemma 3.1.1 and Proposition 3.1.3 do not need the hypothesis that $\hat{G}$ acts on $\hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}}$ trivially. So the same proof works here. $\square$

In general, the functor $\hat{T}$ here may not be essentially surjective. Here is an example provided by Xavier Caruso.

Example 4.2.3. Assume $p > 2$. Let $K = \mathbb{Q}_p(\zeta_p)$, $\pi = \zeta_p - 1$, $K_1 = K(\sqrt[p]{\zeta_p - 1})$, $\pi_1 = \sqrt[p]{\zeta_p - 1}$ and $E(u)$ the Eisenstein polynomial of π. Then $E(u^p)$ is the Eisenstein polynomial of π_1. Let $\mathfrak{M}$ be rank-2 finite free Kisin module of $E(u^p)$-height 1 given by $\varphi(e_1, e_2) = (e_1, e_2) \begin{pmatrix} 1 & u \\ 0 & E(u^p) \end{pmatrix}$, where $\{e_1, e_2\}$ forms a basis of $\mathfrak{M}$. By Theorem (0.4) in [Kis06], there exists a unique crystalline representation V over K_1 such that $V|_{G_\infty} = T_{\mathfrak{S}_1}(\mathfrak{M})$, where $\mathfrak{S}_1 = W(k)[\![u]\!]$ is regarded as a subring of $W(R)$ via $u \mapsto [\underline{\pi}_1]$ and $\underline{\pi}_1 = (\pi_n)_{n \geq 1} \in R$. Now set $W = \mathrm{Ind}^G_{G_1} V$ where $G_1 = \mathrm{Gal}(\overline{K}/K_1)$. Select $\tau \in G$ such that the image of τ in $\mathrm{Gal}(K_1/K)$ is a generator. For any $\sigma \in G$, we denote by V_σ the G_1-action on V given by $\sigma^{-1} g \sigma$. Since $W|_{G_1} \simeq \bigoplus_{i=0}^{p-1} V_{\tau^i}$, W is crystalline over K_1. Now we claim that W is not of finite $E(u)$-height. We prove the claim by contradiction. Assume that there exists a Kisin module $\mathfrak{N}$ such that $T_{\mathfrak{S}}(\mathfrak{N}) \simeq W|_{G_\infty}$, where $\mathfrak{S}$ is regarded as a subring of $W(R)$ via $u \to [\underline{\pi}]$. Then $\mathfrak{S} \subset \mathfrak{S}_1 \subset W(R)$ via $[\underline{\pi}] = [\underline{\pi}_1]^p$. It is easy to check that $\mathfrak{N}' := \mathfrak{N} \otimes_{\mathfrak{S}} \mathfrak{S}_1$ is a Kisin module of $\mathfrak{S}_1$-level and $T_{\mathfrak{S}_1}(\mathfrak{N}') = T_{\mathfrak{S}}(\mathfrak{N})$ as $\mathbb{Z}_p[G_\infty]$-modules. Now note that $W|_{G_\infty} = \bigoplus_{i=0}^{p-1} V_{\tau^i}|_{G_\infty}$ and the functor $T_{\mathfrak{S}_1}$ is fully faithful, so we have $\mathfrak{N}' \simeq \bigoplus_{i=0}^{p-1} \mathfrak{M}_i$ with $\mathfrak{M}_0 \simeq \mathfrak{M}$. Let $\{e_1, \dots, e_{2p}\}$ be a basis of $\bigoplus_{i=0}^{p-1} \mathfrak{M}_i$ such that

$$\varphi(e_1, \dots, e_{2p}) = (e_1, \dots, e_{2p}) \begin{pmatrix} 1 & u & & & \\ 0 & E(u^p) & & & \\ & & * & \cdots & * \\ & & \vdots & \ddots & \vdots \\ & & * & \cdots & * \end{pmatrix} \tag{4.2.2}$$

Select an $\mathfrak{S}$-basis $\{f_1, \dots, f_{2p}\}$ of $\mathfrak{N}$ and write $\varphi(f_1, \dots, f_{2p}) = (f_1, \dots, f_{2p})A$ and $(e_1, \dots, e_{2p}) = (f_1, \dots, f_{2p})X$, where A, X are matrices with coefficients in $\mathfrak{S}$ and $\mathfrak{S}_1$ respectively. Now write A' the matrix in (4.2.2). We have $XA' = A\varphi(X)$. Let x_{ij} denote (i,j) entry of X. Note that all coefficients in $A\varphi(X)$ are in $W(k)[\![[\underline{\pi}_1]^p]\!]$. For $i = 1, \dots, 2p$, we have $x_{i1} \in W(k)[\![[\underline{\pi}_1]^p]\!]$ and $ux_{i1} + E(u^p)x_{i2} \in W(k)[\![[\underline{\pi}_1]^p]\!]$, where $u = [\underline{\pi}_1]$. Since $E(u^p) \equiv u^{pe} \mod p$, we easily see that $x_{i1} \equiv uy_{i1} \mod p$ for $y_{i1} \in \mathfrak{S}_1$. But this contradicts that $X \mod p$ is invertible.

4.3. **The proof of Theorem 2.3.1 when $p=2$.** Let us finish the proof of Theorem 2.3.1 (2) by proving $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ is semi-stable when $p=2$. From the proof of Theorem 4.2.2, we have seen that $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ is semi-stable over K_l, and $\bar{D}$ defined in (4.2.1) is just $D_{\mathrm{st}}(V^\vee) = (B^+_{\mathrm{st}} \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\mathfrak{M}))^{G_l}$. Thus G acts on $\bar{D}$ and it suffices to show that the G-action on $\bar{D}$ is trivial. Let $e_1, \dots, e_d$ be a basis of D. Then $(\bar{e}_1, \dots, \bar{e}_d) = (e_1, \dots, e_d) \sum\limits_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \otimes (N/p^l)^i$ is a basis of $\bar{D}$. For any $g \in G$, the fact that G acts on $\bar{D}$ implies that there exists a matrix B with coefficients in K_0 such that $g(\bar{e}_1, \dots, \bar{e}_d) = (\bar{e}_1, \dots, \bar{e}_d)B = (e_1, \dots, e_d) \sum\limits_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \otimes (N/p^l)^i B$. On the other hand, recall that $\log([\underline{\epsilon}(g)]) = \alpha(g)t$ for an $\alpha(g) \in \mathbb{Z}_p$. So $g(\mathfrak{u}) = g(\log(u)) = \mathfrak{u} + \alpha(g)t$. Write $g(e_1, \dots, e_d) = (e_1, \dots, e_d)A(t)$. We have

$$\begin{aligned} g(\bar{e}_1, \dots, \bar{e}_d) &= g(e_1, \dots, e_d) g\left(\sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \otimes (N/p^l)^i\right) \\ &= (e_1, \dots, e_d)A(t) \sum_{i=0}^{\infty} \gamma_i(\mathfrak{u} + \alpha(g)t) \otimes (N/p^l)^i \\ &= (e_1, \dots, e_d)A(t) \left(\sum_{i=0}^{\infty} \gamma_i(\alpha(g)t) \otimes (N/p^l)^i\right)\left(\sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \otimes (N/p^l)^i\right). \end{aligned}$$

Hence we have

$$\left(\sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \otimes (N/p^l)^i\right) B = A(t) \left(\sum_{i=0}^{\infty} \gamma_i(\alpha(g)t) \otimes (N/p^l)^i\right)\left(\sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \otimes (N/p^l)^i\right).$$

Note that $\mathfrak{u}$ is not algebraic over B_{cris}. We can regard the elements on both sides of the above equation as polynomials in $B^+_{\mathrm{cris}}[\mathfrak{u}]$. Comparing the degree 0 terms, we get $A(t)\left(\sum\limits_{i=0}^{\infty} \gamma_i(\alpha(g)t) \otimes (N/p^l)^i\right) = B$. Hence $A(0) = B$. But g acts on $\hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}}$ trivially so that g acts on $D = \mathcal{D}/(I_+ S_{K_0})\mathcal{D}$ trivially, and we see $\mathrm{Id} = A(0) = B$. Hence G acts on $\bar{D}$ trivially and V is semi-stable over K.

We end this section by asking the following natural questions, inspired by Theorem 4.2.2 and Example 4.2.1.

Question 4.3.1. (1) Notations as in Theorem 4.2.2. What is the essential image of $\hat{T}$?
(2) Let V be a representation of finite $E(u)$-height. Is V potentially semi-stable?

Errata for [Liu07]

As indicated in Remark 3.1.5, there is a gap in §7.1 of [Liu07], on the 3rd to last line of page 668 of the published version. To close this gap, we need to prove that $A(0) = \mathrm{Id}$. In this section, references are to [Liu07], unless explicitly stated otherwise. We will freely use the notations of those sections of that paper to which we refer. Recall the situation in the beginning of §7.1: we have a finite free $\mathbb{Z}_p$-representation T such that for all n, $T_n := T/p^nT \simeq L_{(n)}/L'_{(n)}$ as $\mathbb{Z}[G]$-modules, where $L'_{(n)} \subset L_{(n)}$ are lattices in a semi-stable representation $V_{(n)}$ with Hodge-Tate weights in $\{0, \dots, r\}$. We have proved that there exist $\mathfrak{M}$, $\mathfrak{L}'_{(n)}$ and $\mathfrak{L}_{(n)} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$

such that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T|_{G_\infty}$ and $\mathfrak{M}_n \simeq \mathfrak{L}'_{(n)}/\mathfrak{L}_{(n)}$ [8], where $\mathfrak{L}_{(n)} \hookrightarrow \mathfrak{L}'_{(n)}$ is induced by the injection $L'_{(n)} \subset L_{(n)}$. Recall that $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ and $\mathcal{R} := \mathcal{R}_{K_0} \cap A_{\mathrm{cris}}$. Proposition 6.1.1 showed that $p^{s_0}\tau(\mathcal{M}) \subset \mathcal{M} \otimes_S \mathcal{R}$, where s_0 is a constant only depending on r.

Recall the ideal I_+A constructed in §2.2 in this note for a subring $A \subset B^+_{\mathrm{cris}}$. Write $\tilde\tau = p^{s_0}\tau$ and $\tilde\tau_n$ the action of $\tilde\tau$ on $\mathcal{M}_n \otimes_S \mathcal{R}$. To show $A(0) = \mathrm{Id}$, it suffices to show that

$$\text{for any fixed } x \in \mathcal{M}, \quad \tilde\tau(x) - p^{s_0}x \in (I_+\mathcal{R})\mathcal{M}.$$

If $\mathcal{M}$ comes from a semi-stable representation, then we easily see the above is true by the formula $\tau(x) = \sum\limits_{i=0}^{\infty} N^i(x) \otimes \gamma_i(t)$, where N is the monodromy operator on $\mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Using this fact for $L_{(n)}$ and $L'_{(n)}$, we get $\tilde\tau'_n(x) - p^{s_0}x \in (I_+\mathcal{R})\mathcal{M}_n$. We write $\tilde\tau'_n$ here because it is not clear that $\tilde\tau_n$ and $\tilde\tau'_n$ are the same on $\mathcal{M}_n \otimes_S \mathcal{R} = \mathfrak{M}_n \otimes_{\varphi,\mathfrak{S}} \mathcal{R}$. On the other hand, as in diagram (6.2.1), consider the morphism $\iota_n : \mathfrak{M}_n \otimes_{\varphi,\mathfrak{S}} \mathcal{R} \longrightarrow (T/p^nT)^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$. Note that ι_n is equivariant for both $\tilde\tau'_n$ and $\tilde\tau_n$. Therefore $\iota_n(\tilde\tau'_n(x) - \tilde\tau_n(x)) = 0$. By Lemma 5.3.4, we have $t^r(\tilde\tau'_n(x) - \tilde\tau_n(x)) = \iota^*_n(\iota_n(\tilde\tau'_n(x) - \tilde\tau_n(x))) = 0$. Now considering $\mathcal{M}_n \otimes_S \mathcal{R}$ as a submodule of $\mathcal{M}_n \otimes_S A_{\mathrm{cris}}$ (it is not hard to see that $\mathcal{R}_n$ injects into $A_{\mathrm{cris}}/p^nA_{\mathrm{cris}}$) and using Lemma 6.1.3, we see that there exists a constant λ such that $\tilde\tau'_n(x) - \tilde\tau_n(x) \in (p^{n-\lambda}A_{\mathrm{cris}})\mathcal{M}_n + I^{[1]}\mathcal{M}_n$ for $n \geq \lambda$. We claim that $I^{[1]} \subset I_+A_{\mathrm{cris}}$. In fact, by Proposition 5.3.1 in [Fon94a], any $y \in I^{[1]}$ can be written in the form $\sum_{i \geq 1} a_i t^{\{i\}}$ with $a_i \in W(R)$. Since $t^{\{i\}} \in I_+A_{\mathrm{cris}}$, we have $y \in I_+A_{\mathrm{cris}}$ and then $I^{[1]} \subset I_+A_{\mathrm{cris}}$. Now for all $n \geq \lambda$, we have

$$\tilde\tau_n(x) - p^{s_0}x = \tilde\tau_n(x) - \tilde\tau'_n(x) + \tilde\tau'_n(x) - p^{s_0}x \in (p^{n-\lambda}A_{\mathrm{cris}})\mathcal{M}_n + (I_+A_{\mathrm{cris}})\mathcal{M}_n$$

on $\mathcal{M}_n \otimes_S A_{\mathrm{cris}}$. Hence $\tilde\tau(x) - p^{s_0}x \in (I_+A_{\mathrm{cris}})\mathcal{M}$ and we are done.

As indicated by Remark 4.1.2 in this note, Lemma 8.0.4 is false. So one has to fix the proof in §8. Now let $K' = K_{p^\infty} \cap K_\infty$. If $K' = K$ then we have the same proof of semi-stability of $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ as the case $p \geq 3$. Now suppose that $K' \neq K$. By Proposition 4.1.5 in this note, we see that $K' = K_1$. Write $K_{1,p^\infty} = \bigcup\limits_{n=1}^{\infty} K_1(\zeta_{2^n})$. Since $K_{1,p^\infty} = K_{p^\infty}$, we have $K_{1,p^\infty} \cap K_\infty = K_1$. Hence V is semi-stable over K_1. Now consider $\tilde K = K(\zeta_4)$. By Proposition 4.1.5 in this note, $\tilde K_{p^\infty} \cap \tilde K_\infty = \tilde K$. So V is semi-stable over $\tilde K$. By Proposition 4.1.5 in this note again, $\zeta_4 \notin K_1$. So $K_1 \neq \tilde K$. Then V is semi-stable over $K = K_1 \cap \tilde K$.

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