MODULARITY OF COMPATIBLE FAMILY OF \( p \)-ADIC REPRESENTATIONS

1. Introduction

This note proves the modularity of certain compatible family of \( l \)-adic Galois representations, via Serre and Kisin’s arguments.

2. Compatible family of \( p \)-adic representations

Following [Tay06], we define that a rank 2 weakly compatible system of \( p \)-adic representations \( \mathcal{R} \) over \( \mathbb{Q} \) is a 5-tuple \( (E, S, \{Q_l(X)\}, \rho_p, \{n_1,n_2\}) \) where

- \( E \) is a number field over \( \mathbb{Q} \);
- \( S \) is a finite set of primes over \( \mathbb{Q} \);
- for each prime \( l \notin S \), \( Q_l(X) \) is a monic degree 2 polynomial in \( E[X] \);
- for each prime \( p \) of \( E \), let \( p \) be the residue characteristic.

\[
\rho_p : G := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(E_p)
\]

is a continuous representation such that, if \( p \notin S \) then \( \rho_p|_{G_p} \) is crystalline, and if \( l \notin S \) and \( l \neq p \) then \( \rho_p \) is unramified at \( l \) and \( Fr_l \) has characteristic polynomial \( Q_l(X) \);

- \( \{n_1,n_2\} \) are integers such that for all primes \( p \) of \( E \) (lying above a prime \( p \)) the representation \( \rho_p|_{G_p} \) has Hodge-Tate weights \( n_1 \) and \( n_2 \).

Lemma 2.1. Either all the \( \rho_p \) is absolutely irreducible or all are absolutely reducible.

Proof. Now suppose that \( \rho_p \) is absolutely reducible and we want to show that for any other \( \lambda \in \text{Spec}(E) \), \( \rho_\lambda \) is also absolutely reducible. Note that there exists a finite extension \( K \) over \( E_p \) such that \( \rho_p \) is reducible. Then there is a vector \( e_1 \) in the underline space \( V' = V \otimes_{E_p} K \) such that \( G \) is stable over \( e_1 \). Let \( \chi_1' \) be the character of \( G \) acts on \( e_1 \), \( \chi_2' \) the character \( G \) acts on \( V'/K \cdot e_1 \). Since \( \chi_i' \) is \( p \)-adic Hodge-Tate (i.e. potentially-semi-stable) character. Using Fontaine’s classification, we can prove that \( \chi_i'|_H \simeq \varepsilon^{n_i'}_{p} \), where \( H \subset I_p \) is an open subgroup. Since \( \rho_p|_{G_p} \) has Hodge-Tate weights \( n_1 \), \( n_2 \). So we have no choice but \( n_i = n_i' \) with \( i = 1, 2 \). Now \( \chi_i = \chi_i' \varepsilon^{n_i} \) is a character of \( G \) such that \( \chi_i \) has ramification at primes in \( S \cup p \). Let \( L \) be the splitting field of \( \chi_i \). We claim that \( L \) must be a finite abelian extension of \( \mathbb{Q} \). Let \( L' \subset L \) be a finite abelian subfield. It suffices that we can bound the conductor of \( L' \). There are two cases: Case I, let \( l \in S \) and \( l \neq p \). Set \( I_{l,L'}^w \) the wild ramification group (i.e., the \( l \)-Sylow subgroup inside the ramification group \( I_{l,L'} \)). Then \( i : I_{l,L'}^w \rightarrow O_{E_p}^* \). We claim that \( I_{l,L'}^w \rightarrow O_{E_p}^*/1+p \). Suppose that \( i(x) \in 1+p \). Note that \( 1+p \) is profinite \( p \)-group, but \( i(x) \) has order \( l \)-power, thus \( i(x) = 1 \) and \( I_{l,L'}^w \rightarrow O_{E_p}^*/1+p \). Thus the order \( I_{l,L'}^w \) is bounded. By [Ser] §4.9 Proposition 9, the conductor at \( l \) is bounded; Now considering the case II, conductor at \( p \). The ramification index at \( p \) is \( |I_p:H| \). Therefore we also bounded the conductor at \( p \).

In conclusion, we can bounded the conductor of \( L' \) and then the splitting field of
\( \chi_i \) is finite. Therefore the images of \( \chi_i \) are finite. Then there exists finite extension \( E'/E \) such that images of \( \chi_i \) are inside \( \mathcal{O}_{E'} \).

Now for any \( \rho_q \) for \( q \neq p \) and \( q \) over rational prime \( q \). Consider the \( q \)-adic representation \( \rho'_q = \epsilon_q \chi_1 + \epsilon_q \chi_2 \) defined over \( E'_q \), where prime \( q' \in \text{Spec}(E') \) over \( q \). Since \( \epsilon_p \) is compatible family of 1-dimensional \( p \)-adic Galois representation, the characteristic polynomial of \( F_{l} \) of \( \rho'_q \) is the same as the that of \( \rho_p \) for almost all primes \( l \). Thus the characteristic polynomial of \( F_{l} \) of \( \rho'_q \) is the same as that of \( \rho_q \) for almost all primes \( l \). Thus by Chebotarev density theorem, the traces of \( \rho'_q \) and \( \rho_q \) are the same. Then \( \rho_q \) is reducible.

\[ \square \]

We call \( \mathcal{R} \) regular if \( n_1 < n_2 \) and \( \det \rho_p(c) = -1 \) for one (and hence all) primes, where \( c \) denotes complex conjugation. Set \( \epsilon = (\epsilon_p) \) the compatible system of \( p \)-adic cyclotomic characters. For any \( i \in \mathbb{Z} \), denote \( \mathcal{R}(i) \) the system \( (\rho_p \epsilon_p^i) \).

**Lemma 2.2.** \( \mathcal{R}(-n_1) \) is weakly compatible system with Hodge-Tate weights 0 and \( n_2 - n_1 \).

**Proof.** It suffices to show that for any \( p \not\in S \) and \( l \not\in S \) and \( l \neq p \). The characteristic polynomial \( f_l(X) \) of \( F_{l} \) is independent of choice of \( p \). Note that \( f_l(X) = \det(IX - \rho_p \epsilon_p^{-n_1}(F_{l})) = \det(IX - l^{-n_1} \rho_p((F_{l}))) = l^{-2n_1} \chi_l^{-1}(X) \). \( \square \)

Now we state the classical theorem on compatible system constructed from modular forms. For any prime \( p \), we fix an embedding \( E \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p \). Let \( k \geq 2 \), \( N \geq 1 \) and \( S_k(\Gamma_1(N), \mathbb{C}) \) the space of cuspidal modular form with weight \( k \) and level \( N \). Suppose that \( f = \sum_{i=1}^{\infty} a_n q^n \) is an eigenform normalized such that \( a_1 = 1 \).

**Theorem 2.3.** Notations as above, then \( E_f = \mathbb{Q}(a_n)_{n \geq 1} \subset \mathbb{C} \) is a number field. Moreover, for any \( \lambda|p \) of \( E_f \), there exists a continuous representation

\[ \rho_{f,\lambda} : G \longrightarrow \text{GL}_2(E_{f,\lambda}) \]

such that

1. \( \rho_{f,\lambda} \) is odd and absolutely irreducible.
2. For any \( l \not\mid Np \), \( \rho_{f,p} \) is unramified over \( l \) and \( \text{tr}(\rho_{f,\lambda}(F_{l})) = a_l \).
3. For any \( \lambda|p \), \( \rho_{f,\lambda}|_{G_p} \) is potential semi-stable with Hodge-Tate weights in \( \{0, k-1\} \). If \( \lambda \not\mid N \), the \( \rho_{f,\lambda}|_{G_p} \) is crystalline.

Let \( \rho_i : G \to \text{GL}_2(E_i), i = 1, 2 \) be a two representations with \( E_i \) finite extensions of \( \mathbb{Q}_p \) (resp. \( \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z} \)). We write \( \rho_1 \sim \rho_2 \) if there exist an finite extension \( E/\mathbb{Q}_p \) (resp. \( E/\mathbb{F}_p \)) such that \( E_i \subset E \) for all \( i = 1, 2 \) and \( \rho_1 \otimes_E E \simeq \rho_2 \otimes_E E \).

**Theorem 2.4.** Let \( \mathcal{R} = (\rho_p) \) be an irreducible regular rank 2 weakly compatible system with weights \( \{0, k\} \). Then there exists an eigenform \( f \) with weight \( k + 1 \) such that for any \( \rho_p \) there exists a prime \( \lambda \) of \( E_f \) satisfying \( \rho_p \sim \rho_{f,\lambda} \).

Note that all \( \rho_p \) in Theorem 2.4 and \( \rho_{f,\lambda} \) here are irreducible. To show they are isomorphic, using Chebotarev’s density theorem, it suffices to show that there exists \( \rho_p, f \) and \( \lambda \) such that \( \text{tr}(\rho_p(F_{l})) = \text{tr}(\rho_{f,\lambda}(F_{l})) \) for almost all \( l \).
Remark 2.5. The assumption on absolutely irreducibility here is equivalent to the following condition (pure weight $k$):

For any $l \notin S$ and for all $i : E \to \mathbb{C}$ the root of $i(Q_l(X))$ have absolute value $\frac{p^k}{2}$.

See Lemma 3.2 below for the proof.

Corollary 2.6. If $d = 1$ then the Scholl representation is modular.

3. The proof of Theorem 2.4

In the proof, we mainly use Serre’s conjecture. So let us first review Serre’s conjecture. In the sequel, for any prime $l$, we use $G_l \subset G$ to denote the decomposition group over prime $l$ and $I_l \subset G_l$ the inertia subgroup. $\mathbb{F}/\mathbb{F}_p$ is always a finite field with characteristic $p$.

3.1. The strong Serre’s conjecture. In this subsection, we recall the precise form of Serre’s conjecture which predicts not only that an odd representation $\bar{\rho} : G \to \text{GL}_2(\mathbb{F})$ arises from a modular form, but also the minimal weight and level of the form.

Let $\omega_i : I_p \to \mathbb{F}_p^\times ; g \mapsto g\left(\frac{p^i-\sqrt{p}}{p^i-1}\right) \mod p$ be the fundamental character of level $i$. We will write $\omega$ for $\omega_1$, which is mod $p$ reduction of $p$-adic cyclotomic character $\epsilon_p$.

Suppose we are given a representation $\bar{\rho}_p : G_p \to \text{GL}_2(\mathbb{F}_p)$. Then $\bar{\rho}_p|I_p$ is either of the form $\left(\begin{array}{cc} \omega^i & 0 \\ 0 & 1 \end{array}\right) \otimes \omega_j$ with $i, j \in \mathbb{Z}$ or $\left(\begin{array}{cc} \omega^i & 0 \\ 0 & \omega^j \end{array}\right) \otimes \omega_j$ for some integers $i, j \in \mathbb{Z}$ and $p + 1 \nmid i$.

When $\bar{\rho}_p|I_p$ is semi-simple, or equivalently tamely ramified, we can always choose $j \in [0, p-2]$ and $i + j \in [1, p-1]$; when $\bar{\rho}_p|I_p$ is wildly ramified $i, j \in [0, p-2]$ can be uniquely determined. We set $k(\bar{\rho}_p) = 1 + i + (p + 1)j$, unless $\bar{\rho}_p|I_p \sim \left(\begin{array}{cc} \omega^i & 0 \\ 0 & 1 \end{array}\right) \otimes \omega_j$, with $\ast$ très ramifié. In this exceptional case, we set $k(\bar{\rho}_p) = (p + 1)(j + 1)$.

For a representation $\bar{\rho} : G \to \text{GL}_2(\mathbb{F})$, we set $k(\bar{\rho}) = k(\bar{\rho}|_{G_p})$ and set

$$N(\bar{\rho}) = \prod_{l \neq p} \text{cond}(\bar{\rho}|_{G_l}),$$

where $\text{cond}(\bar{\rho}|_{G_l})$ is the Artin conductor of $\bar{\rho}|_{G_l}$. Let $V$ be the underlying space of $\bar{\rho}$, then $\text{cond}(\bar{\rho}|_{G_l}) = l^{n_l}$ where

$$n_l = \sum_{i=0}^{\infty} \frac{1}{(G_0 : G_i)} \dim(V/V^{G_i})$$

(3.1)

where $G_i \subset G_0 = I_l$ are the ramification subgroups.

Theorem 3.1 (Serre’s conjecture). Let $\bar{\rho} : G \to \text{GL}_2(\mathbb{F})$ be odd and absolutely irreducible. Then there exists an eigenform $f$ with weight $k(\bar{\rho})$ and level $N(\bar{\rho})$ such that $\bar{\rho} \sim \bar{\rho}_{f, \lambda}$. 

3.2. The proof of Theorem 2.4. Now we use Theorem 3.1 to prove the Theorem 2.4. We claim that there exist infinite many primes \( p_i \in \text{Spec}(E) \) such that

1. \( k(\bar{\rho}_{p_i}) = k + 1 \) for all \( i \)
2. The set \( \{N(\bar{\rho}_{p_i})\} \) is bounded.
3. For all \( i \), \( \bar{\rho}_{p_i} \) is absolutely irreducible.

Let us first accept claim and prove Theorem 2.4. Suppose the set \( \{p_i\} \) does exist. By Theorem 3.1, for any \( i \), there exists an eigenform \( f_i \in S_{k+1}(\Gamma_1(N(\bar{\rho}_{p_i})), \mathbb{C}) \) such that \( \bar{\rho}_{p_i} \sim \bar{\rho}_{f_i, \lambda_i} \). Select an \( N \) such that \( N(\bar{\rho}_{p_i})|N \) for all \( i \). We see that \( f_i \) are eigenforms in \( S_{k+1}(\Gamma_1(N), \mathbb{C}) \), which is a finite dimensional \( \mathbb{C} \)-space. So there are only finitely many normalized eigenforms. Therefore, there exists an eigenform \( f \) such that \( f_i = f \) for infinitely many \( i \). Without loss of generality, we assume that \( f_i = f \) for all \( i \).

Now for any fix prime \( l \notin S \), let \( a_l \) be the coefficient of \( X \) in \( Q_l(X) \). Since \( \{\rho_{p_i}\} \) is compatible, for any \( p_i \neq l \), \( a_l = \text{tr}(\rho_{p_i}(F_l)) \). On the other hand, set \( b_l \) the \( l \)-th Fourier coefficient of \( f \). Then we have \( b_l = \text{tr}(\rho_{f, \lambda_i}(F_l)) \). Choose a Galois extension \( F/\mathbb{Q} \) which contains \( E \) and \( E_f \). Without loss of generality, we can assume our embedding \( \iota : E \hookrightarrow F \hookrightarrow \bar{\mathbb{Q}}_l \) in a way such that \( p_i = O_E \cap m \) with \( m \) the maximal ideal of \( O_{\bar{\mathbb{Q}}_l} \). Then \( \lambda_i \) is determined an embedding \( \sigma_i : E_f \hookrightarrow F \hookrightarrow \bar{\mathbb{Q}}_l \). But there are only finitely many embeddings \( E_f \hookrightarrow F \) here, so there must be an embedding \( \sigma \) such that \( \sigma_i = \sigma \) for infinitely many \( i \). Without loss of generality, we can assume that \( \sigma = \sigma_i \) for all \( i \), and we embed \( E_f \hookrightarrow \bar{\mathbb{Q}}_l \) and \( \lambda_i = E_f \cap m \). Set \( q_i = F \cap m \).

Now since \( \bar{\rho}_{p_i} \sim \bar{\rho}_{f, \lambda_i} \). Thus \( a_l = b_l \mod q_i \) for all \( i \). Since there are infinitely many \( i \), we see that \( a_l = b_l \) for all \( l \notin S \). This prove Theorem 2.4.

3.3. The proof of the claim. The first two claims are not hard, while the last one need more work.

For any \( p \notin S \) and \( p > k + 1 \), we claim that for any \( p \mid p, k(\bar{\rho}_p) = k + 1 \). In fact, since \( \rho_p|G_E \) is crystalline and Hodge-Tate weights are 0, \( k \) with \( k \leq p - 2 \). The one can use Fontaine-Messing theory on strongly divisible lattices in filtered \( \varphi \)-modules to compute the reduction of such crystalline representations. Let \( T \) be a lattice in \( \rho_p|G_E \) and denote \( \bar{T} \) the reduction of \( T \). There are two cases:

Case I: \( T \) is irreducible, then \( \bar{T}|_{I_p} \otimes \mathbb{F}_p \sim \begin{pmatrix} \omega_2^k & 0 \\ 0 & \omega_2^{pk} \end{pmatrix} \). So \( k(\bar{\rho}_p) = k + 1 \).

Case II: \( T \) is reducible, then \( \bar{T}|_{I_p} \otimes \mathbb{F}_p \sim \begin{pmatrix} \omega_k^k & \ast \\ 0 & 1 \end{pmatrix} \). In this case if \( k > 1 \), then we see that \( k(\bar{\rho}_p) = k + 1 \). For \( k = 1 \), we must eliminate the case that \( T|_{I_p} \) is très ramifié. And this case can be eliminated by some explicit computations.

Now let us bound the conductors of \( \bar{\rho}_{p_i} \). We claim that there exist infinitely many primes \( p_i \in \text{Spec}(E) \) such that the set \( \{N(\bar{\rho}_{p_i})\} \) is bounded. First note that for any \( l \notin S \) and \( l \neq p \), then \( \rho_p \) is unramified at \( l \). Therefore the conductor \( N(\bar{\rho}_{p}) \) only consists those primes in \( S \). For any \( l \in S \), let \( n_l \) be the integer defined in (3.1). Let \( F \) be the splitting field of \( \rho_p \) and \( G_0 \) the inertia subgroup at \( l \) inside \( \text{Gal}(F/\mathbb{Q}) \) and \( G_1 \) the \( l \)-Sylow subgroup (i.e., teh wild inertia). Assume that the residue field \( [k_p : \mathbb{F}_p] = g \). Note that \( g \leq [E : \mathbb{Q}] \). Since that \( G_0 \hookrightarrow \text{GL}_2(k_p) \). Thus we have \( l^{m_l} = \#(G_1)/(p^g - 1)(p^g - p^g) \). Thus \( (p^g - 1)(p^g - 1) = 0 \mod l^{m_l} \). Select an integer \( a_l \) sufficient large (depending on \( l \) and \( g \)) such that there exists a \( b_i \in \mathbb{Z}/l^{a_l}\mathbb{Z} \) satisfying \( b_i^2 \neq 1 \mod l^{a_l} \). Therefore for any primes \( p \) such that
If there exists infinitely many primes \( p \) such that \( p \equiv b_i \mod l^{m_i}, \ l^{m_i} \nmid (p^{2g} - 1)(p^{2g} - p^9) \). Thus for any primes \( p \) above \( p, \ m_i < a_i \). Now by [Ser] §4.9 Proposition 9, so \( n_l \leq 2(a_i + \frac{1}{l^{a_i}}) \). Now by Chinese reminder theorem, we can select infinitely many primes \( p \) such that \( p = b_i \mod l^{m_i} \). Thus there exists infinity many primes \( p_i \) over those \( p \) such that \( \{N(\hat{\rho}_{p_i})\} \) is bounded.

**Lemma 3.2.** Let \( \mathcal{R} = (\rho_p) \) be a regular rank 2 weakly compatible family with weights \( \{0, k\} \). Then \( \mathcal{R} \) is absolutely irreducible if and only the following condition holds:

For any \( l \not\in S \) and for all \( i : E \to \mathbb{C} \) the root of \( i(Q_l(X)) \) have absolute value \( l^{k/2} \).

Moreover, there exists for infinitely many prime \( p_i \) such that \( \hat{\rho}_{p_i} \) is absolutely irreducible.

**Proof.** If \( \mathcal{R} \) is irreducible, by main theorem of [Tay06], we see that for any \( l \not\in S \) and for all \( i : E \to \mathbb{C} \) the root of \( i(Q_l(X)) \) have absolute value \( l^{k/2} \).

Now by the proof of the Lemma, we see that there exists a set \( S' \) of infinitely many primes \( p_i \) such that \( k(\hat{\rho}_{p_i}) = k + 1 \) and the set of \( \{N(\hat{\rho}_{p_i})\} \) is bounded. Now we claim that there are only finitely many primes \( p_i \in S' \) such that \( \hat{\rho}_{p_i} \) is absolutely reducible. Now suppose that there exists a subset \( S'' \subset S' \) of infinitely many primes such that for any \( p_i \in S'' \) \( \hat{\rho}_{p_i} \) is absolutely reducible. We would like to derive a contradiction.

Note that for any \( p \geq k + 2, \ p \not\in S \) and \( p|p \), then \( T := \rho_p|_{G_{\mathbb{P}}} \) is crystalline. As discussed in the beginning of this subsection, we have 2 cases of reduction, where the first case is absolutely irreducible. So if \( \hat{\rho}_{p} \) is absolutely reducible, then we must have the second case. Therefore we have

\[
\hat{\rho}_{p} \otimes \overline{\mathbb{F}}_p \sim \left( \begin{array}{cc} \chi_1 \omega^k & \ast \\ 0 & \chi_2 \end{array} \right)
\]

where \( \chi_1, \chi_2 \) are characters unramified at \( p \). It is easy to see that the conductor \( N(\chi_j)|N(\hat{\rho}_{p}) \) for \( j = 1, 2 \). We can lift \( \chi_j \) to \( \hat{\chi}_j : G \to \hat{\mathbb{Z}}^* \) with the same conductor. For any \( p_i \in S'' \), write \( \hat{\chi}_j^{(i)} \) for characters attached to \( \hat{\rho}_{p_i} \). Since the set of \( \{N(\hat{\rho}_{p_i}), \ i \in S'\} \) is bounded, conductors \( \hat{\chi}_j^{(i)} \) are bounded. So there are only finitely many \( \hat{\chi}_j^{(i)} \). Therefore, without loss of generality, we can assume that \( \hat{\chi}_j = \hat{\chi}_j^{(i)} \) for all \( i \).

Now select a finite Galois extension \( F \) such that \( F \) contains \( E \) and all values of \( \hat{\chi}_1 \) and \( \hat{\chi}_2 \). Using the same trick in the second paragraph of the proof of Theorem 2.4, we can assume \( p_i \in S'' \) are all primes in \( \mathcal{O}_F \). Now we claim that for any fixed prime \( q|\mathcal{O}_F \) and \( q \in S'' \), the semi-simplification \( \rho_q \) is \( \left( \begin{array}{cc} \epsilon_q \chi_1 & 0 \\ 0 & \chi_2 \end{array} \right) \), where \( \epsilon_q \) is the \( q \)-adic cyclotomic character. It suffices to prove that their traces coincides for almost all primes. For any \( l \not\in S \) and \( q | l \), let \( a_l = \text{tr}(\rho_q(Fr_l)) \). For any \( p_i | l, \ p_i|p \) and \( p_i \in S'' \), we also have \( a_l = \text{tr}(\rho_{p_i}(Fr_l)) \). But \( \hat{\rho}_{p_i} \) is reducible and has shape (3.2). So we have

\[
a_l = \text{tr}(\hat{\chi}_1 \epsilon_q^k + \hat{\chi}_2)(Fr_l)) \mod p_i.
\]

Note that \( (\epsilon_p) \) is compatible system and there are infinitely many \( p_i \). We have \( a_l = \text{tr}(\hat{\chi}_1 \epsilon_p^k + \hat{\chi}_2)(Fr_l)) \) and the semi-simplification \( \rho_q \) is just \( \left( \begin{array}{cc} \epsilon_q \chi_1 & 0 \\ 0 & \chi_2 \end{array} \right) \). This is impossible because the roots of \( Q_l(X) \) have absolute value \( l^{k/2} \) for all \( l \). So except
finitely many primes in $S'$, $\bar{\rho}_p$ are absolutely irreducible. \qed

4. Totally real case

Now let us extend some results of the above to the totally real case. Let $F$ be a totally real field with $[F : \mathbb{Q}] = g$. Assume that $F$ is Galois. A regular\(^1\) rank 2 weakly compatible system of $\lambda$-adic representations $R_F$ over $F$ is a 5-tuple $(E, S, \{Q_l(X)\}, \rho_\lambda, \{n_1, n_2\})$ where

- $E$ is a number field over $F$;
- $S$ is a finite set of primes over $F$;
- for each prime $l \notin S$ of $F$, $Q_l(X)$ is a monic degree 2 polynomial in $E[X]$;
- for each prime $\lambda$ of $E$, let $p$ be the residue characteristic and $p := \lambda \cap O_F$.

$$
\rho_\lambda : G_E := \text{Gal}(\mathcal{F}/F) \longrightarrow GL_2(E_\lambda)
$$

is a continuous representation such that, if $p \notin S$ then $\rho_\lambda|_{G_p}$ is crystalline with the sets Hodge-Tate weights being $\{n_1, n_2\}$, and if $l \notin S$ then $\rho_\lambda|_{G_l}$ is unramified at $l$ and the Frobenius $\text{Fr}_l$ has the characteristic polynomial $Q_l(X)$.

- $n_1 < n_2$ and $\det\rho_p(c) = -1$ for one (and hence all) primes, where $c$ denotes complex conjugation.

**Remark 4.1.** In general, one should consider the case that the sets of Hodge-Tate weights are $\{n_1, \ldots, n_g\}$ in $R$ instead of just $\{n_1, n_2\}$. But the definition of more general treatment is much more complicated. For our ad hoc concern of this note, we only consider the simpler case.

We call the compatible family $R_F$ has a semi-descent to $\mathbb{Q}$ if for any prime $\lambda|p$ over $O_E$ there exists a $\lambda$-adic representation $\rho_\lambda' : G_\mathbb{Q} \rightarrow GL_2(L_\lambda)$ and $L_\lambda$ is a finite extension of $E_\lambda$ such that $\rho_\lambda'|_{G_p} \sim \rho_\lambda$. Of course, if there exists a weakly compatible family $R_\mathbb{Q} = (\rho_\lambda')$ over $\mathbb{Q}$ such that $\rho_\lambda'|_{G_p} \sim \rho_\lambda$. Then we see that $R_\mathbb{Q}$ is a quasi-descent of $R_F$. In this case, we call that $R_F$ has a descent to $\mathbb{Q}$.

**Remark 4.2.** In general, the descent of $R_F$ is not unique.

**Theorem 4.3.** Definitions as the above, let $R_F = (\rho_\lambda)$ be a regular rank 2 weakly compatible family over $F$. Suppose that For any $l \notin S$ and for all $i : E \rightarrow \mathbb{C}$ the root of $i(Q_l(X))$ have absolute value $\frac{l^{2g-2n_2}}{n_2}$. Then $R_F$ has a semi-descent to $\mathbb{Q}$ if and only if $R_F$ has a descent $R_\mathbb{Q}$ to $\mathbb{Q}$. In this case there exists a modular form $f$ such that $R_\mathbb{Q}$ comes from $f$.

Of course if we know that $R_F$ can be descent to a weakly compatible family $R_\mathbb{Q}$ over $\mathbb{Q}$. Then the last statement is just the consequence of Theorem 2.4. But the proof of the Theorem actually circle around: we first prove that $R_F$ comes from a modular form $f$ over $\mathbb{Q}$. Then construct the descent $R_\mathbb{Q}$ via $f$. Without loss of generality, we assume that $n_1 = 0$ and $n_2 = k$ from now on. To carry out the first step, we use the similar strategy of the proof of Theorem 2.4. So the following Lemma is crucial.

**Lemma 4.4.** There exists a set $S' \subset \text{Spec}(O_E)$ of infinitely many primes such that

1. $k(\bar{\rho}_\lambda') = k + 1$ for any $\lambda \in S'$.

\(^1\)need a better name here
(2) \( \{N(\bar{\rho}_\lambda)|\lambda \in S\} \) is bounded.

(3) \( \bar{\rho}_\lambda \) is absolutely irreducible for any \( \lambda \in S' \).

**Proof.** To prove the lemma, we first construct a set \( \bar{S} \) of infinitely many primes \( \lambda \) such that \( \bar{S} \) satisfies the first two requirements. For any prime \( \lambda | p \) in \( \mathcal{O}_F \), write \( p = \lambda \cap F \). We claim that if \( p \geq k + 2 \) and \( p \notin S \cup \{p, p|\Delta_F\} \) where \( \Delta_F \) is the discriminant of \( F \), then \( k(\bar{\rho}_\lambda) = k + 1 \). To see this, for any such \( \lambda \), consider \( \rho'_\lambda|_{G_{\bar{q}_p}} \). Note that \( F_p \) is unramified extension over \( \mathbb{Q}_p \), and \( \rho'_\lambda|_{G_{\bar{q}_p}} \) is crystalline with Hodge-Tate weights \( \{0, k\} \), so is \( \rho'_\lambda|_{G_{\bar{q}_p}} \). So the discussion of weights in the beginning of §3.3 applies, we get \( k(\bar{\rho}_\lambda) = k + 1 \).

Let us bound the conductor of \( \bar{\rho}_\lambda \). We only need show there exist infinitely many primes \( \lambda_i \) such that \( m_i \) defined in (3.1) are bounded for those rational primes \( l \) such that there is a prime \( l | l \) and \( l \in \mathfrak{S} \). Let \( L' \) be the splitting field of \( \bar{\rho}_\lambda \) and \( G_0 \) the inertia subgroup at \( l \) inside \( \text{Gal}(L'/\mathbb{Q}) \) and \( G_1 \) the \( l \)-Sylow subgroup (i.e., the wild inertia subgroup). Write \( m_i = \#(G_1) \). By [Ser] §4.9 Proposition 9, it suffices to bound \( m_i \) for certain \( \lambda_i \). Now consider \( \rho_\lambda \). Let \( L \) be the splitting field of \( \rho_\lambda \) and select \( l \in \text{Spec}(\mathcal{O}_F) \) is a prime over \( l \). We also define \( G_0, G_1, m_i \) (respect to \( l \)) respectively. We claim that \( m_i \leq \log(l) + m_l \) and hence it suffices to bound \( m_l \). To see the claim, note that \( \rho_\lambda \sim \rho'_\lambda|_{G_{\bar{q}_p}} \). Then \( L' = LF \) and \( |L'| = |L : \mathbb{Q}| = g \). Consequently \( |G_0 : G_0| = l \) and \( |G_1 : G_1| \) \leq g. Hence \( m_l \leq m_l + \log(l) \). Now we can bound \( m_l \) just like \( F = \mathbb{Q} \) case: Assume that the residue field \( k_\lambda : \mathbb{F}_p \) is \( h \). Note that \( h \leq [E : \mathbb{Q}] \). Since that \( G_0 \leftarrow \text{GL}_2(k_\lambda) \). Thus we have \( m_l = \#(G_1)(p^{2h} - 1)(p^{2h} - p^h) \). Thus \( p^{2h} - 1 = p^h - 1 \) is \( \text{mod} \ l \). Select an integer \( a_l \) large (depending on \( l \) and \( h \)) such that there exists a \( b_l \in (\mathbb{Z}/l^{a_l}\mathbb{Z})^* \) satisfying \( b_l^h \neq 1 \) \( \text{mod} \ l \). Therefore for any primes \( p \) such that \( p \equiv b_l \) \( \text{mod} \ l^{a_l} \), \( p^{2h} - 1 \neq p^h - 1 \) \( \text{mod} \ l^{a_l} \). Now by Chinese remainder theorem, there exists infinitely many primes \( \lambda_i \) over \( p_i \) such that \( p_i \equiv b_i \) \( \text{mod} \ l \). Hence \( \{N(\bar{\rho}_\lambda)\} \) are bounded.

Now let us treat the last claim. By the proof above, we see that there exists a set \( S' \) of infinitely many primes \( \lambda_i \) such that \( k(\bar{\rho}_\lambda) = k + 1 \) and the set \( \{N(\bar{\rho}_\lambda)\} \) is bounded. Now we claim that there are only finitely many primes \( \lambda_i \in S' \) such that \( \bar{\rho}_\lambda \) (Caution: not only \( \bar{\rho}_\lambda \) ) is absolutely reducible. Now suppose that there exists a subset \( S'' \subset S' \) of infinitely many primes such that for any \( \lambda \in S'' \), \( \bar{\rho}_\lambda \) is absolutely reducible. We would like to derive a contradiction.

Note that for any rational prime \( p \geq k + 2 \), \( p \notin S \) and \( \lambda | p \) (recall that \( p = \lambda \cap \mathcal{O}_F \)), then \( \rho'_\lambda|_{G_{\bar{q}_p}} \) is crystalline. If \( p \notin \Delta_F \) then we see that \( \rho'_\lambda|_{G_{\bar{q}_p}} \) is crystalline. Note we have two types of reductions for \( \rho'_\lambda|_{G_{\bar{q}_p}} \) as discussed in the beginning of §3.3. Either \( \bar{\rho}_\lambda|_{I_p} \otimes \mathbb{F}_p \simeq \begin{pmatrix} \omega_p^k & 0 \\ 0 & \omega_p^k \end{pmatrix} \) or \( \bar{\rho}_\lambda|_{I_p} \otimes \mathbb{F}_p \sim \begin{pmatrix} \omega_k & * \\ 0 & 1 \end{pmatrix} \). If \( \bar{\rho}_\lambda \) has the first reduction type. Note that \( F_p \) is unramified over \( \mathbb{Q}_p \). Then \( \bar{\rho}_\lambda|_{I_p} \sim \bar{\rho}_\lambda|_{I_p} \) and hence \( \bar{\rho}_\lambda \) must be absolutely irreducible. So if \( \lambda \) is in \( S'' \), \( \bar{\rho}_\lambda|_{I_p} \) must have the second reduction type. Hence we have \( \bar{\rho}_\lambda \otimes \mathbb{F}_p \sim \begin{pmatrix} \chi_{1}\omega^k & * \\ 0 & \chi_2 \end{pmatrix} \) where \( \chi_1, \chi_2 \) are characters unramified at all primes over \( p \). Restricted to \( G_F \), we have

\[
\bar{\rho}_\lambda \otimes \mathbb{F}_p \sim \begin{pmatrix} \chi_{1}\omega^k & * \\ 0 & \chi_2 \end{pmatrix}
\]

where \( \chi_1, \chi_2 \) are characters unramified at all primes over \( p \). It is easy to see that the conductor \( N(\chi_j)|N(\bar{\rho}_\lambda) \) for \( j = 1, 2 \). We can lift \( \chi_j \) to \( \hat{\chi}_j : G_F \rightarrow \mathbb{Z}^* \) with the
same conductor. For any \( \lambda_i \in S'' \), write \( \hat{\chi}_j^{(i)} \) for characters attached to \( \bar{\rho}_{\lambda_i} \). Since the set of \( \{N(\bar{\rho}_{\lambda_i}), \ i \in S'\} \) is bounded, conductors \( \hat{\chi}_j^{(i)} \) are bounded. So there are only finitely many \( \hat{\chi}_j^{(i)} \). Therefore, without loss of generality, we can assume that \( \hat{\chi}_j = \hat{\chi}_j^{(i)} \) for all \( i \).

Now select a finite Galois extension \( L \) such that \( L \) contains \( E \) and all values of \( \hat{\chi}_1 \) and \( \hat{\chi}_2 \). Using the same trick in the second paragraph of the proof of Theorem 2.4, we can assume \( \lambda_i \in S'' \) are all primes in \( \mathcal{O}_L \). Now we claim that for any fixed prime \( q|q \) of \( \mathcal{O}_L \) and \( q \in S'' \), the semi-simplification \( \rho_q \) is \( \begin{pmatrix} \epsilon_q \hat{\chi}_1 & 0 \\ 0 & \hat{\chi}_2 \end{pmatrix} \), where \( \epsilon_q \) is the \( q \)-adic cyclotomic character. It suffices to prove that their traces coincides for almost all primes. For any \( l|l \notin S \) (\( l \) is a rational prime) and \( q \not| l \), let \( a_l = \text{tr}(\rho_q(\text{Fr}_l)) \). For any \( \lambda_i \not| l \), \( \lambda_i|p \) and \( \lambda_i \in S'' \), we also have \( a_l = \text{tr}(\rho_{\lambda_i}(\text{Fr}_l)) \). But \( \bar{\rho}_{\lambda_i} \) is reducible and has shape (4.1). So we have

\[
a_l = \text{tr}(\langle \hat{\chi}_1 \epsilon_q^k + \hat{\chi}_2 \rangle(\text{Fr}_l)) \mod \lambda_i.
\]

Note that \( (\epsilon_q) \) is compatible system and there are infinitely many \( \lambda_i \). We have \( a_l = \text{tr}(\langle \hat{\chi}_1 \epsilon_q^k + \hat{\chi}_2 \rangle(\text{Fr}_l)) \) and the semi-simplification \( \rho_q \) is just \( \begin{pmatrix} \epsilon_q \hat{\chi}_1 & 0 \\ 0 & \hat{\chi}_2 \end{pmatrix} \). This is impossible because the roots of \( Q_l(X) \) have absolute value \( l^{k/2} \) for all \( l \). So except finitely many primes in \( S' \), \( \bar{\rho}_{\lambda_i} \) are absolutely irreducible. 

**Proof of Theorem 4.3.** By Lemma 4.4, we see that \( \bar{\rho}_{\lambda_i}^j \) is absolutely irreducible for all \( \lambda_i \in S' \). Hence by Serre’s conjecture (Theorem 3.1), there exists eigenform \( f_i' \in S_{k+1}(\Gamma_1(N(\bar{\rho}_{\lambda_i}^j)), \mathbb{C}) \) such that \( \bar{\rho}_{f_i',\alpha_i}^j \sim \bar{\rho}_{\lambda_i}^j \), where \( \alpha_i \) is a prime of \( E_f \), over \( p \). Let \( f_i \) be the base change \( f_i' \) to \( F \). We have \( \bar{\rho}_{f_i,\alpha_i} \sim \bar{\rho}_{\lambda_i} \). Since \( f_i \) has the same weight and level as those \( f_i' \). Now we just copy the remaining proof of Theorem 2.4. And we see that there exists \( f_i = f \) such that \( \rho_f \) gives the compatible family \( \mathcal{R}_F \). Consequently, picking and \( f_i' \) such that \( f_i = f \), then \( f_i' \) gives compatible family \( \mathcal{R}_Q \) over \( \mathbb{Q} \) which is a descent of \( \mathcal{R}_F \). 

5. Descend a \( p \)-adic Galois representation

In this section, let us discuss an intrinsic condition to descend a \( p \)-adic Galois representation. Let \( F, K \) be number fields and \( F/K \) Galois, \( g = [F : \mathbb{Q}] \), \( E/\mathbb{Q}_p \) a finite extension and \( \rho : G_F := \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{Aut}_E(V) \) a \( p \)-adic Galois representation. We say \( \rho \) satisfies *quasi-descent* condition if the following holds:

There exists a finite set of primes \( S_p \subset \text{Spec}(\mathcal{O}_K) \) such that

1. \( S_p \) contains all ramified primes of \( F/K \) and \( \rho \).
2. For any primes \( l \) of \( \mathcal{O}_K \) such that \( l \not\in S_p \cup \{p\} \), write \( l = (\varphi_1 \varphi_2 \ldots \varphi_m)^{\infty} \) in \( \mathcal{O}_F \), then

\[
\det(\lambda I - \rho(\text{Fr}_{\varphi_i})) = \det(\lambda I - \rho(\text{Fr}_{\varphi_j})) \text{ for any } i, j = 1, \ldots, m.
\]

Apparently, suppose that there exists a \( p \)-adic representation \( \rho' : G_K \to \text{Aut}_E(V') \) such that \( \rho'|_{G_F} \sim \rho \) then \( \rho \) satisfies the quasi-descent condition. In this case, we call \( \rho' \) is a descent of \( \rho \). Conversely, we have the following question:
Question 5.1. Is that true that ρ satisfies the quasi-descent condition if ρ has a descent?

Remark 5.1. In general, V and V′ are not defined in the same coefficients fields, even if the descent exists. Here is an example: let p = 3, select a cyclic extension F/Q with degree g such that there exists a g-th roots of unity not in Z_3, and E/Z_3 a finite extension such that g-th roots of unity are all in O_E. Consider the Galois character ρ′ : G_Q → Gal(F/Q) → O_E by sending the generator of Gal(F/Q) to the g-th root of unity. ρ = ρ′|_{G_F} is a trivial representation. So ρ can be defined over Q_3. This example also shows that given ρ the descent may not be unique. In fact, any character of Gal(F/Q) can be a descent of ρ.

However we can classify all descents of ρ in some nice situations. Let σ ∈ G_K, in the sequel, we write ρ′′ : G_F → Aut_E(V′′) be another descent of ρ. We need show that there exists a χ ∈ H such that ρ′′ ∼ ρ′ ⊗ χ. Without loss of generality, we can assume that all representations here are finite dimensional \overline{Q}_p-spaces. Note that both α′ := ρ′(σ) and α′′ := ρ′′(σ) induce isomorphisms ρ → ρ″. Hence (α″)(α′)−1 ∈ Aut_{\overline{Q}_p[G_F]}(ρ).

Proposition 5.2. Suppose that F/K is cyclic and ρ is absolutely irreducible. Assume that ρ′ is a descent of ρ. Let H be the group of all E-characters of Gal(F/K). Then {ρ′ ⊗ χ | χ ∈ H} exhausts all possible descents of ρ.

Proof. Let σ ∈ G_K such that σ is a generator of Gal(F/K). Assume that ρ″ : G_K → Aut_E(V″) be another descent of ρ. Select a σ ∈ G_K such that σ is a generator of Gal(F/K). The quasi-descent conditions implies that there exists an isomorphism f_σ : V^σ → V. Then it is easy to check that (f_σ)^g : V^{σ^g} → V is an isomorphism. On the other hand, f′ := ρ(σ^g) induces to isomorphism V^{σ^g} → V. So (f′)(f_σ)^{−g} : V → V is inside Aut_{\overline{Q}_p[G_F]}(V), which is \overline{Q}_p by the absolutely irreducibility of ρ. So there exists an a ∈ \overline{Q}_p such that (f_σ)^g a^g = f′. After replace f_σ by f_σ a, we can assume that f_σ = f′ = ρ(σ^g).

Now let us construct ρ′ as following: For any τ ∈ G_K, τ can be written uniquely τ = σ^m β with 0 ≤ m < g and β ∈ G_F. Set ρ′(τ) = (f_σ)^m ρ(β). Now it suffices to check that ρ′(τ σ) = ρ′(τ_1)ρ(τ_2). Now write τ_1 = σ^m β_1. We have ρ′(τ_1)ρ′(τ_2) = ((f_σ)^m ρ(β_1))((f_σ)^m ρ(β_2)) = (f_σ)^m_1 m_2 ((f_σ)^{-m_2} ρ(β_1)(f_σ)^{m_2} ρ(β_2)).

Now let us gives a partial answer to Question 5.1.

Proposition 5.3. Assume that F/K is cyclic and ρ is absolutely irreducible. Then Conjecture 5.1 is true.

Proof. Without loss of generality, we assume that E = \overline{Q}_p. Select a σ ∈ G_K such that σ is a generator of Gal(F/K). The quasi-descent conditions implies that there exists an isomorphism f_σ : V^σ → V. Then it is easy to check that (f_σ)^g : V^{σ^g} → V is an isomorphism. On the other hand, f′ := ρ(σ^g) induces to isomorphism V^{σ^g} → V. So (f′)(f_σ)^{−g} : V → V is inside Aut_{\overline{Q}_p[G_F]}(V), which is \overline{Q}_p by the absolutely irreducibility of ρ. So there exists an a ∈ \overline{Q}_p such that (f_σ)^g a^g = f′. After replace f_σ by f_σ a, we can assume that f_σ = f′ = ρ(σ^g).
One the other hand,
\begin{equation}
\tau_1 \tau_2 = \sigma^{m_1 + m_2} \sigma^{-m_2} \beta_1 \sigma^{m_2} \beta_2 = \sigma^{m'} \sigma^{qg} (\sigma^{-m_2} \beta_1 \sigma^{m_2}) \beta_2
\end{equation}
where \(m_1 + m_2 = m' + qq\) with \(0 \leq m' < g\). Hence
\begin{equation}
\rho'(\tau_1 \tau_2) = (f_\sigma)^{m'} \rho(\sigma^{qg}) \rho(\sigma^{-m_2} \beta_1 \sigma^{m_2}) \rho(\beta_2)
\end{equation}

Now we need to check \((f_\sigma)^{m_1 + m_2} = (f_\sigma)^{m'} \rho(\sigma^{qg})\) and \((f_\sigma)^{-m_2} \rho(\beta_1)(f_\sigma)^{m_2} = \rho(\sigma^{-m_2} \beta_1 \sigma^{m_2})\). But these follows the facts that \(f_\sigma^g = \rho(\sigma^g)\) and \(f_\sigma\) is an isomorphism \(V^\sigma \to V\).

Now let \(R_F = (\rho_\lambda)\) is a regular weakly compatible system over a Galois totally real field \(F\) as defined in \S4. Suppose that one representation \(\rho_\beta\) satisfies the quasi-descent condition. Since \((\rho_\lambda)\) is a compatible family, all \(\rho_\lambda\) satisfies the quasi-descent. Now combining Proposition 5.3 and theorem 4.3 together, we have

**Theorem 5.4.** Let \(R_F = (\rho_\lambda)\) be a regular weakly compatible system of \(\lambda\)-adic Galois representations over a totally real field \(F\). Suppose the following holds:

1. \(F/\mathbb{Q}\) is cyclic.
2. One of \(\rho_\lambda\) satisfies the quasi-descent condition.
3. For any \(I \not\in S\) and for all \(i : E \to \mathbb{C}\) the root of \(i(Q_I(X))\) have absolute value \(l^{2-n_2}\).
4. Except finitely many primes, \(\rho_\lambda\) are absolutely irreducible.

**Remark 5.5.** It seems that one can relax (1) to case that \(F\) is solvable and (4) seems to be removable. Furthermore, (3) also can be removed if one can extend the main results of [Tay06] to totally real field. But all these seems need some non-trivial efforts.

**Corollary 5.6.** Let \(\rho\) be a \(p\)-adic Galois representation of \(G_\mathbb{Q}\). Suppose that there exists a totally real field \(F/\mathbb{Q}\) such that \(\rho\) restricted to \(G_F\) comes from a Hilbert modular form \(f\) over \(F\) and \(F/\mathbb{Q}\) is cyclic. Then \(\rho\) comes from a modular form.

**Proof.** \(f\) gives arise a regular compatible family \(R_F\) which satisfies (3) and (4) in the above theorem. Since \(\rho\) is a descent of \(\rho|_{G_F}\), \(\rho\) satisfies the quasi-descent condition. Then the corollary follows.

**References**
