ON AUTOMORPHY OF CERTAIN GALOIS REPRESENTATIONS OF GO_4-TYPE

TONG LIU AND JIU-KANG YU

with an appendix by Liang Xiao

To Prof. Wen-Ching Winnie Li

Abstract. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$ be a continuous representation. We prove (potential) automorphy theorems for certain types of ρ. Our results include several cases in which the Hodge-Tate weights are irregular. Finally, we prove (potential) automorphy for certain compatible systems of representations of GO_4-type, which includes certain compatible systems constructed from Scholl motives.

Contents

1991 *Mathematics Subject Classification.* Primary 14F30,14L05.

Key words and phrases. Galois representations, automorphy.

This materials is based upon work supported by National Science Foundation under agreement No. DMS-0635607. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

1. Introduction

Let L be a number field and G_L denote the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$. For any place v of L, we denote L_v the v-completion of L and denote G_{L_v} a decomposition subgroup of G_L, which is isomorphic to $\mathrm{Gal}(\overline{L}_v/L_v)$. We fix an isomorphism $\iota_v : \overline{L}_v \simeq \mathbb{C}$ for each v throughout this paper.

The aim of this note is to prove several automorphy theorems of certain p-adic Galois representations $\rho : G_{\mathbb{Q}} \to \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$. Recall that

$$\mathrm{GO}_n(\overline{\mathbb{Q}}_p) := \{X \in \mathrm{GL}_n(\overline{\mathbb{Q}}_p) | XX^T = r(X)I_n,\ r(X) \in \overline{\mathbb{Q}}_p^\times\}.$$

It is obvious that $r : \mathrm{GO}_n \to \mathbb{G}_m$ is a character and it is called the *multiplier* of GO_n. Let SGO_4 be the neutral component of I_n in GO_4, which can be characterized by $\mathrm{SGO}_4(\overline{\mathbb{Q}}_p) = \{X \in \mathrm{GO}_4(\overline{\mathbb{Q}}_p) | (\det \cdot r^{-2})(X) = 1\}$. See §2.1 for more details for GO_4 and SGO_4. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be a continuous representation. We denote by $\mathrm{HT}(\rho)$ for the set of Hodge-Tate weights.

Theorem 1.0.1. *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_p)$ be an irreducible continuous representation. Assume the following*

1. $\rho(G_{\mathbb{Q}}) \subset \mathrm{SGO}_4(\overline{\mathbb{Q}}_p)$*;*
2. *ρ is unramified almost everywhere;*
3. *The eigenvalues of the complex conjugation c on ρ are $-1, -1, 1, 1$ and $r(c) = 1$;*
4. *$\rho|_{G_{\mathbb{Q}_p}}$ is crystalline and $\mathrm{HT}(\rho) = \{0, m, n, m+n\}$ with $0 < m, n < \frac{p}{2}$;*

Then ρ is automorphic, that is, ρ arises from an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$.

Note that by duality if $\rho(G) \subset \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$ and $\rho|_{G_{\mathbb{Q}_p}}$ is Hodge-Tate then after twisting by a power of the cyclotomic character, $\mathrm{HT}(\rho) = \{0, m, n, m+n\}$ for non-negative integers m and n.

Applying the above theorem to a compatible system $\{E, S, \{Q_l(X)\}, \{\rho_\lambda\}, \mathbf{v}\}$ in the sense of *weakly* compatible system (with minor modifications, see §4.2) in [BLGGT11, §5.1], we obtain the following result which also includes the case $\mathrm{HT}(\rho) = \{0, 0, m, m\}$ for $m > 0$ (this case needs special treatment).

Theorem 1.0.2. *Let $\mathcal{R} := \{E, S, \{Q_l(X)\}, \{\rho_\lambda\}, \mathbf{v} = \{0, m, n, m+n\}\}$ be a compatible system of Galois representations of $G_{\mathbb{Q}}$. Assume the following:*

1. *ρ_λ is absolutely irreducible for each prime λ over $\mathcal{O}_E$;*
2. *$\rho_\lambda(G_{\mathbb{Q}}) \subset \mathrm{SGO}_4(\overline{\mathbb{Q}}_p)$ for each λ;*
3. *The multipliers $\{r(\rho_\lambda)\}$ forms a compatible system, $r(\rho_\lambda)(c) = 1$and the eigenvalues of the complex conjugation c on ρ_λ are $-1, -1, 1, 1$ for a prime λ (hence for all λ);*
4. $\max\{m, n\} > 0$.

Then $\mathcal{R}$ is automorphic.

Let us state one of our applications of the above theorem. In [Sch85] and [Sch96], Scholl constructed a compatible system of $2d$-dimensional p-adic Galois representations $\{\rho_p\}$ of G_L attached to the space of cusp forms $S_k(\Gamma, \mathbb{C})$, where $d = \dim_{\mathbb{C}} S_k(\Gamma, \mathbb{C})$, $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a *noncongruence* subgroup and L is the field of definition for the curve defined by $\mathfrak{H}/\Gamma$ with $\mathfrak{H} \subset \mathbb{C}$ the upper half plane. It has been proved by Scholl that the family $\{\rho_p\}$ is motivic with Hodge-Tate weights $\{0, 0, k-1, k-1\}$. Furthermore, if $k \geq 3$ is odd then $\rho_p(G_L) \subset \mathrm{GO}_{2d}(\overline{\mathbb{Q}}_p)$.

Corollary 1.0.3. *Let $\{\rho_p : G_{\mathbb{Q}} \to \mathrm{GL}_4(\mathbb{Q}_p)\}$ be the compatible system of Galois representations attached to a Scholl motive. Assume that $k \geq 3$ is odd and*

(1) *ρ_p is absolutely irreducible for each p;*
(2) *$\rho_p(G_{\mathbb{Q}}) \subset \mathrm{SGO}_4(\overline{\mathbb{Q}}_p)$ (via a conjugation)*

Then ρ is automorphic.

In general, some conditions are needed to guarantee $\rho_p(G_{\mathbb{Q}}) \subset \mathrm{SGO}_4(\overline{\mathbb{Q}}_p)$. The following is an example.

Corollary 1.0.4. *Assume that $k \geq 3$ is odd and ρ_p is absolutely irreducible for each p. If the Scholl motive admits a real multiplication of a real quadratic field then ρ_p is automorphic.*

If we only assume that $\rho(G_{\mathbb{Q}}) \subset \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$ instead of $\mathrm{SGO}_4(\overline{\mathbb{Q}}_p)$ then we can only prove potential automorphy for certain ρ. It is easy to check that the Galois character $(\det r^{-2}) \circ \rho$ only takes value ± 1. Let F be the (necessarily real under the hypotheses of the following theorem, see §3.2) quadratic extension determined by this character.

Theorem 1.0.5. *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$ be an irreducible continuous representation. Assume the following*

(1) *$p \geq 6$ and p is unramified over F;*
(2) *ρ is unramified almost everywhere;*
(3) *The eigenvalues of the complex conjugation c on ρ are $-1, -1, 1, 1$ and $r(c) = 1$;*
(4) *$\rho|_{G_{\mathbb{Q}_p}}$ is crystalline and $\mathrm{HT}(\rho) = \{0, m, n, m+n\}$ with $0 < m, n < p-1$ and $m \equiv n \mod 2$;*
(5) *The reduction $\bar{\rho}|_{G_{F(\zeta_p)}}$ is irreducible.*

Then ρ is potentially automorphic, that is, there exists a totally real field F' such that $\rho|_{G_{F'}}$ arises from an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{F'})$.

We remark that our theorem is (basically) covered by Theorem C in [BLGGT11], except the case that $\mathrm{HT}(\rho) = \{0, m, m, 2m\}$ with $m \geq 1$. The Hodge-Tate weights in this case are *irregular*, namely, Hodge-Tate weights of ρ are *not* distinct. Note that [BLGGT11] only (essentially) discussed the regular cases. Also we are able to prove the potential automorphy of compatible systems in this case.

Theorem 1.0.6. *Let $\mathcal{R} = \{E, S, \{Q_l(X)\}, \{\rho_\lambda\}, \mathbf{v} = \{0, m, m, 2m\}\}$ be a compatible system of Galois representations of $G_{\mathbb{Q}}$. Assume the following:*

(1) *ρ_λ is absolutely irreducible for each prime λ over $\mathcal{O}_E$;*
(2) *$\rho_\lambda(G_{\mathbb{Q}}) \subset \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$ for each λ;*
(3) *The multipliers $\{r(\rho_\lambda)\}$ forms a compatible system, $r(\rho_\lambda)(c) = 1$ and the eigenvalues of the complex conjugation c on ρ_λ are $-1, -1, 1, 1$ for a prime λ (hence for all λ);*
(4) *$m \geq 1$.*

Then the system $\mathcal{R}$ is potentially automorphic.

Now let us discuss the strategy and plan in this paper. We use very similar strategy as that used in [Ram02]. As the structure of SGO_4 is very close to $\mathrm{GL}_2 \times \mathrm{GL}_2$, by Tate's theorem, we are able to show in §2 that there exist 2-dimensional Galois representations $\rho_i : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ for $i = 0, 1$ such that $\rho|_{G_F} \simeq \rho_0 \otimes \rho_1$. §3 is devoted to proving that each ρ_i satisfies the hypothesis of an automrphy theorem for GL_2 (see Theorem 4.2.1). That is, ρ_i is unramifed all most everywhere, odd and crystalline at primes over p. It turns out that the most technical part is the local properties of ρ_i at primes over p. Note that ρ_i is constructed purely abstractly (the existence guaranteed by the vanishing of group cohomology), so it is not known a priori that ρ_i is even Hodge-Tate, though $\rho_0 \otimes \rho_1$ is crystalline. Luckily, in §3.3 we can modify Di Matteo's theorem in [DM13] (also see Liang Xiao's new approach in the appendix) to show that there exists a character χ such that $\rho_0 \otimes \chi$ and $\rho_1 \otimes \chi^{-1}$ are crystalline at primes above p, under the condition $m \equiv n \mod 2$. But this covers the most interesting case (the irregular weight case). Then in §4, we are able to use modularity or potential automorphy lifting theorems for GL_2 to prove each ρ_i is modular or potentially automorphic. Hence the (potential) automorphy of ρ follows the main theorem of [Ram00]. In the end, we treat Theorem 1.0.2 in the case that $\mathrm{HT}(\rho_\lambda) = \{0, 0, m, m\}$ and discuss its application to certain compatible systems of representations coming from Scholl motives.

When this paper was nearly complete, we found that our parer has some overlap with the preprints [Cal], [Con] and [Pat13]. More precisely, the trick that the automorphy of representation ρ of GO_4-type can be reduced to the automorphy of 2-dimensional representations via tensor product and Ramakrishnan's theorem was also known and used in [Cal] and [Pat13]. Questions 3.3.1 in §3.3 is formulated differently (see Question 3.3.2) and in a general setting in [Con], [Pat13], [Pat14] and some answers to these questions are provided. These answers almost cover results obtained in §3.3 (see Remark 3.3.3 for details). Here we remarks that our method in §3.3 is totally elementary and self-contained. Also our paper focuses on the automorphy of certain Galois representation with irregular weights (e.g., Galois representations arising from Shcoll motives) and these have not been discussed by these papers.

Acknowledgement: It is a pleasure to thank Laurent Berger, Brian Conrad, Frank Calegari, Stefan Patrikis, Richard Taylor and Liang Xiao for very useful conversations and correspondence. We are grateful to the anonymous referee for lots of useful comments, which have greatly improved the exposition of this paper. The first author would like to thank IAS for support to complete this paper. He is partially supported by NSF grant DMS-0901360 and Sloan Fellowship.

2. Representation to GO_4 and tensor product

2.1. **Preliminary on GO_4.** We recall some basic definitions and properties of GO_4 and refer readers to §1 in [Ram02] for more details. Let E be an algebraically closed field and V a $2n$-dimensional E-vector space with a quadratic form Q. Then the associated orthogonal similitude group is $\mathrm{GO}(V, Q) :=$

$$\{g \in \mathrm{GL}(V) \mid \exists r(g) \in E^\times \text{ such that } Q(gv) = r(g)Q(v), \quad \forall v \in V\}.$$

The character $r : \mathrm{GO}(V, Q) \to E^\times$ is the *similitude multiplier* or simply *multiplier*. When Q is non-degenerate, it is easy to see that the character $\nu := \det \cdot r^{-n} : \mathrm{GO}(V, Q) \to E^\times$ maps surjectively onto $\mu_2(E)$. The kernel of ν, denoted by

$\mathrm{SGO}(V,Q)(E)$, is called the *special orthogonal similitude group.* It is exactly the neutral component group of $\mathrm{GO}(V,Q)$. It is clear that if Q, Q' are non-degenerate then $\mathrm{GO}(V,Q)$ is conjugate to $\mathrm{GO}(V,Q')$ in $\mathrm{GL}(V)$. Therefore, we sometimes suppress V and Q and denote $\mathrm{GO}(V,Q)$ by $\mathrm{GO}_{2n}(E)$, $\mathrm{SGO}(V,Q)$ by $\mathrm{SGO}_{2n}(E)$.

Let $W = E^2$ be the space on which $\mathrm{GL}_2(E)$ acts. Then $V \simeq W \otimes W^*$ and $\beta'(A,B)$ is simply $A \otimes (B^{-1})^*$. It follows that if we put $\beta(A,B) = A \otimes B$, then there is a non-degenrate quadratic form Q on $V := W \otimes W$, unique up to a multiplicative scalar, such that the following is a short exact sequence

$$1 \to E^\times \to \mathrm{GL}_2(E) \times \mathrm{GL}_2(E) \xrightarrow{\beta} \mathrm{SGO}(V,Q) \to 1,$$

where the first map is $c \mapsto (cI_2, c^{-1}I_2)$. We have $r(\beta(A,B)) = \det(A)\det(B)$.

2.2. **Lifting projective representations.** The following theorem has been proved in [Con, §5] (in more general situations). Here we reproduce the proof for being self-contained.

Theorem 2.2.1. *Let H be a connected reductive group over $\overline{\mathbb{Q}}_p$ and let S be a torus contained in the center of H. Let $\tilde{\rho} : G_F \to H(\overline{\mathbb{Q}}_p)/S(\overline{\mathbb{Q}}_p)$ be a continuous homomorphism. Then there exists a continuous homomorphism $\rho : G_F \to H(\overline{\mathbb{Q}}_p)$ lifting $\tilde{\rho}$, that is, the composition $G_F \xrightarrow{\rho} H(\overline{\mathbb{Q}}_p) \to H(\overline{\mathbb{Q}}_p)/S(\overline{\mathbb{Q}}_p)$ is $\tilde{\rho}$.*

Proof. We remark that here $H(\overline{\mathbb{Q}}_p)$ is endowed with the p-adic topology. It is well-known that the theorem is also true when the discrete topology is used instead. In that case, the obstruction to lift lies in $\mathrm{H}^2(G_F, S(\overline{\mathbb{Q}}_p)_{\mathrm{disc}}) = 0$ by a theorem of Tate [Ser77, §6.5], where $S(\overline{\mathbb{Q}}_p)_{\mathrm{disc}}$ is $S(\overline{\mathbb{Q}}_p)$ endowed with the discrete topology. The case here with p-adic topology is probably well-known too. It is based on the following variant of the above argument, which was explained to us by C.S. Rajan when $H = \mathrm{GL}_n$.

Let Z° be the neutral component of the center of H, and let H' be the derived group of H. Then Z° is a torus and there exists a subtorus S' of Z° such that $Z^\circ = S \times S'$. Put $J = H'S'$. Then $A = J \cap S$ is a finite group such that $J/A \simeq H/S$.

The key claim is that for any 2-cocyle c of G_F with values in $A(\overline{\mathbb{Q}}_p)$, there exists a finite subgroup A' of S containing A such that c becomes trivial in $\mathrm{H}^2(G_F, A'(\overline{\mathbb{Q}}_p))$. Indeed, write $T := S(\overline{\mathbb{Q}}_p)_{\mathrm{disc}}$ and consider the exact sequence

$$\mathrm{H}^1(G_F, T) \to \mathrm{H}^1(G_F, T/A(\overline{\mathbb{Q}}_p)) \to \mathrm{H}^2(G_F, A(\overline{\mathbb{Q}}_p)) \to \mathrm{H}^2(G_F, T)$$

induced by the short exact sequence $0 \to A(\overline{\mathbb{Q}}_p) \to T \to T/A(\overline{\mathbb{Q}}_p) \to 0$. Since c becomes trivial in $\mathrm{H}^2(G_F, T)$ by Tate's theorem, c can be lift to $\tilde{c} \in \mathrm{H}^1(G_F, T/A(\overline{\mathbb{Q}}_p)) = \mathrm{Hom}(G_F, T/A(\overline{\mathbb{Q}}_p))$. As G_F is profinite, $\tilde{c}$ has finite image: it takes values in A' for some A' containing A. The claim follows immediately.

The obstruction to lifting $\tilde{\rho}$ to a continuous homomorphism $G_F \to J(\overline{\mathbb{Q}}_p)$ is an element in $\mathrm{H}^2(G_F, A(\overline{\mathbb{Q}}_p))$. Let c be a 2-cocycle representing this element and let A' be a finite subgroup of S containing A. Then the image of c in $\mathrm{H}^2(G_F, A'(\overline{\mathbb{Q}}_p))$ is exactly the obstruction to lift $\tilde{\rho}$ to a continuous homomorphism from G_F to $J'(\overline{\mathbb{Q}}_p)$, where J' is the subgroup generated by J and A' (notice that $J'/A' \simeq J/A \simeq H/S$). By the key claim, we can choose A' to make the obstruction vanish. The theorem follows immediately. $\square$

Corollary 2.2.2. *For any $n \geq 1$, every continuous projective representation $\tilde{\rho}$: $G_F \to \mathrm{PGL}_n(\overline{\mathbb{Q}}_p)$ lifts to a continuous linear representation $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$.*

Corollary 2.2.3. *For every continuous homomorphism $\tilde{\rho} : G_F \to \mathrm{SGO}_4(\overline{\mathbb{Q}}_p)$, there exists a continuous homomorphism $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p) \times \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that $\beta \circ \rho = \tilde{\rho}$.*

3. Properties of tensor factors

In this section, we consider a tensor product $\rho_F = \rho_0 \otimes \rho_1$ of finite-dimensional $\overline{\mathbb{Q}}_p$-representations of G_F, where F is a number field. Let d_i be the degree of ρ_i. In this section, we study how ρ_i (or rather a suitable twist of ρ_i by a character) inherits properties of ρ_F required in the Fontaine-Mazur conjecture. We will soon specialize to the following situation. Start with $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_p)$ with $\rho(G_{\mathbb{Q}}) \subset \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$. Let $G_F = \rho^{-1}(\mathrm{SGO}_4(\overline{\mathbb{Q}}_p))$ so that $[F : \mathbb{Q}] \leq 2$. Then Corollary 2.2.3 implies that $\rho_F := \rho|_{G_F}$ is the tensor product of two 2-dimensional representations ρ_0, ρ_1 of G_F.

3.1. **The unramified almost everywhere property.**

The following proposition has been proved in [Con]. Here we include the proof of proposition for the convenience of readers.

Proposition 3.1.1. *If $\rho_F = \rho_0 \otimes \rho_1$ is unramified almost everywhere then ρ_i is unramified almost everywhere for $i = 0, 1$.*

Proof. By §2 in [Ski09], we may assume that $\rho_i(G_F) \subset \mathrm{GL}_{d_i}(\mathcal{O}_E)$ with E a finite extension of $\mathbb{Q}_p$. Choose an open subgroup J of $\mathrm{GL}_{d_i}(\mathcal{O}_E)$ small enough such that J is a torsion-free pro-p-group. Let L/F be a finite Galois extension such that $\rho_i(G_L) \subset J$. It suffices to show that $\rho_i|_{G_L}$ is unramified almost everywhere. Indeed we claim that $\rho_i|G_{L_v}$ is unramified for all $v \nmid p$ such that $\rho_F|_{G_{L_v}}$ is unramified.

Let $l \neq p$ be the residue characteristic of v and l^m be the cardinality of the the residue field of v. The image under ρ_i of the wild inertia subgroup P_v of $D_v := G_{L_v}$, being both pro-l and pro-p, is necessarily trivial. Thus $\rho_i|_{D_v}$ factors through the quotient D_v/P_v. It is well-known that D_v/P_v is topologically generated by two elements F and T satisfying $FTF^{-1} = T^{l^m}$, and T (topologically) generates the inertia subgroup modulo P_v. By assumption, $\rho(T) = 1$ and hence $\rho_i(T)$ is in the center of $\mathrm{GL}_{d_i}(E)$. This forces $\rho_i(T)$ to be of finite order dividing $l^m - 1$. Since $\rho_i(T)$ lies in the torsion-free group J, we must have $\rho_i(T) = 1$. This proves that $\rho_i|_{D_v}$ is unramified. □

3.2. **Properties at the archimedean places.**

Suppose that $\rho_F := \rho|_{G_F} = \rho_0 \otimes \rho_1$ comes from $\rho : G_{\mathbb{Q}} \to \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$. Let $c \in G_{\mathbb{Q}}$ be a complex conjugation. We further assume that $\rho(c)$ has eigenvalues $1, 1, -1, -1$ and $r \circ \rho(c) = 1$, i.e. $r \circ \rho$ is even.

Since $(\det \cdot r^{-2})(\rho(c)) = 1$, we conclude that c lies in G_F and F is totally real. From $\det(\rho_0)(c) \det(\rho_1)(c) = r \circ \rho(c) = 1$ we deduce $\det \rho_0(c) = \det \rho_1(c)$. If $\det \rho_0(c) = \det \rho_1(c) = 1$, the eigenvalues of $\rho(c)$ would be the same sign repeated 4 times. Therefore, we must have $\det \rho_0(c) = \det \rho_1(c) = -1$. Since this holds for any complex conjugation in G_F, we conclude that ρ_0 and ρ_1 are both totally odd.

3.3. **p-adic Hodge theoretical properties.** We refer to [Ber04] for the definitions and basic properties of Hodge-Tate, de Rham, crystalline, ..., representations and constructions of period rings like B_{st}. Let K be a finite extension of $\mathbb{Q}_p$ and V a finite dimensional $\mathbb{Q}_p$-vector space with continuous $\mathbb{Q}_p$-linear $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$-action. We always denote $(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)}$ by $D_{\mathrm{st}}(V)$.

If $\rho_F = \rho_0 \otimes \rho_1$ is assumed to be crystalline, ρ_i may not be crystalline. In fact, we have that $\rho_F = (\rho_0 \otimes \chi) \otimes (\rho_1 \otimes \chi^{-1})$ for any character χ of G_F. As χ can be non-Hodge-Tate, ρ_i can be non-Hodge-Tate. This leads the following natural question:

Question 3.3.1. *Let F be a number field. Assume that $\rho_i : G_F \to \mathrm{GL}_{d_i}(\overline{\mathbb{Q}}_p)$, $i = 0, 1$ are continuous Galois representations such that $\rho_0 \otimes \rho_1$ is unramified almost everywhere and is semi-stable (resp. crystalline) at G_{F_v} for each prime $v \mid p$. When does there exist a character $\chi : G_F \to \overline{\mathbb{Q}}_p^\times$ such that $\rho_0 \otimes \chi$ and $\rho_1 \otimes \chi^{-1}$ are unramified almost everywhere and are semi-stable (resp. crystalline) at G_{F_v} for each prime $v \mid p$?*

Indeed it is not hard to see the above question is equivalent to the following:

Question 3.3.2. *Suppose that $\rho : G_F \to \mathrm{PGL}_n(\overline{\mathbb{Q}}_p)$ is a continuous representation such that ρ is unramifed almost everywhere and $\rho|_{G_{F_v}}$ is semi-stable (resp. crystalline) for each prime $v \mid p$. Is there a lift $\tilde{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ of ρ so that $\tilde{\rho}$ is unramifed almost everywhere and $\tilde{\rho}|_{G_{F_v}}$ is semi-stable (resp. crystalline) for each prime $v \mid p$?*

Remark 3.3.3. As explained in the end of the introduction, Question 3.3.2 (formulated in a more general setting) is studied in [Con], [Pat13] and [Pat14] and some answers have been provided (see Proposition 5.3, 6.5 in [Con], Theorem 3.2.10, Corollary 3.2.12 in [Pat13], Proposition 5.5 in [Pat14]). The aim of this section is to provide answers to Question 3.3.1 via an elementary and self-contained argument, though many of our results here have been covered by Conrad and Patrikis's results in a more general settings (Proposition 3.3.4 is in [Con], Corollary 3.3.8 and Theorem 3.3.9 are proved in [Pat14]).

The following proposition reduces the question to the existence of a character χ such that $\rho_0 \otimes \chi$ is Hodge-Tate at each $v|p$.

Proposition 3.3.4. *The character χ in Question 3.3.1 exists if and only if there exists a character $\chi' : G_F \to \overline{\mathbb{Q}}_p^\times$ such that $\rho_0 \otimes \chi'$ is Hodge-Tate at G_{F_v} for each prime $v \mid p$.*

Proof of Proposition 3.3.4. After replacing ρ_0 and ρ_1 by $\rho \otimes \chi'$ and $\rho_1 \otimes (\chi')^{-1}$, we can assume that ρ_0 and ρ_1 are Hogde-Tate at each primes $v|p$. By Theorem A.0.1 in the appendix, for each prime $v|p$, ρ_0 and ρ_1 are De Rham at v. By the well-known fact that being De Rham implies that potential semi-stability, we see that ρ_0 and ρ_1 are potentially semi-stable at each prime $v|p$. By Lemma 3.3.5 and Corollary 3.3.6 below, for each prime $v|p$ there exists a local character χ_v of *finite* image such that $\rho_0|_{G_{F_v}} \otimes \chi_v$ and $\rho_1|_{G_{F_v}} \otimes \chi_v^{-1}$ are semi-stable. Applying Lemma 4.1.1 in [CHT08], there exists a finite character $\chi : G_F \to \overline{\mathbb{Q}}_p^\times$ such that $\chi|_{G_{F_v}} = \chi_v$ for all primes $v|p$, and then χ is the desired character. If $\rho_0 \otimes \rho_1$ is further crystalline at each

v then the monodromy operator N on $D_{\mathrm{st}}((\rho_0 \otimes \rho_1)|_{G_{F_v}})$ is 0. As $N = N_0 \otimes N_1$ where N_i is the monodromy operator on $D_{\mathrm{st}}((\rho_0 \otimes \chi)|_{G_{F_v}})$ and $D_{\mathrm{st}}((\rho_1 \otimes \chi^{-1})|_{G_{F_v}})$ respectively, we conclude that $N_i = 0$ for $i = 0, 1$ and hence $\rho_0 \otimes \chi$ and $\rho_1 \otimes \chi^{-1}$ are crystalline at each $v|p$. □

Let K, E be finite extensions of $\mathbb{Q}_p$ and $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. Assume that V is a finite dimensional E-vector space with an E-linear action of G_K such that V is a potentially semi-stable representation. We may assume that V is semi-stable over K', which is Galois over K. Then the Galois group $\Gamma = \mathrm{Gal}(K'/K)$ acts on $D_{\mathrm{st}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/K')}$ semi-linearly. Since V is an E-representations, $D_{\mathrm{st}}(V)$ is an $E \otimes_{\mathbb{Q}_p} K'_0$-module where $K'_0 = W(k')[1/p]$ and k' is the residue field of $\mathcal{O}_{K'}$. Luckily, one can show that $D_{\mathrm{st}}(V)$ is finite $E \otimes_{\mathbb{Q}_p} K'_0$-free (see Lemma 2.1 in [Sav05]). Note there is a Frobenius action φ on $D_{\mathrm{st}}(V)$ such that

- φ and Γ acts on the $E_{K'_0}$-module $D_{\mathrm{st}}(V)$ E-linearly and K'_0-semi-linearly, where $E_{K'_0} := E \otimes_{\mathbb{Q}_p} K'_0$;
- φ and the action of Γ commute.

Now assume that V_i are representations of G_K satisfying the assumptions on V in the above paragraph. Let Γ^I be the inertia subgroup of Γ.

Lemma 3.3.5. *If $V_1 \otimes_E V_2$ are semi-stable then there exists a finite abelian extension K' of K such that both V_i are semi-stable on K'.*

Proof. Without loss of generality, we can assume that Γ acts on $D_{\mathrm{st}}(V_i)$ faithfully. Let $\gamma \in \Gamma^I$ and γ_1, γ_2 denote the matrices of γ acting on $D_{\mathrm{st}}(V_1)$, $D_{\mathrm{st}}(V_2)$ respectively, which are finite free $E_{K'_0}$-modules. Since $\gamma_1 \otimes \gamma_2$ is the identity matrix, there exists a $c(\gamma) \in E^\times_{K'_0}$ such that $\gamma_1 = c(\gamma) I_{d_1}$ and $\gamma_2 = c(\gamma)^{-1} I_{d_2}$ where $d_i = \dim_E(V_i)$ for $i = 1, 2$. Since φ and the action of Γ commutes, writing A for the matrix of φ, the fact that $\varphi\gamma = \gamma\varphi$ implies $A\varphi(\gamma_1) = \gamma_1 A$. As γ_1 is a scalar matrix and A is invertible, we see that $\varphi(c(\gamma)) = c(\gamma)$. Hence $c(\gamma) \in E^\times$. Now for any g in Γ and $\gamma \in \Gamma^I$, we claim that $g\gamma = \gamma g$. In fact, let g_1 and γ_1 denote the matrices of g and γ for a fixed basis $e_1, \dots, e_d$. Note that $\gamma_1 = c(\gamma) I_d$ with $c(\gamma) \in E^\times$. We get

$$\gamma g(e_1, \dots, e_d) = (e_1, \dots, e_d)\gamma_1 g_1 = (e_1, \dots, e_d) g_1 \gamma_1 = g\gamma(e_1, \dots, e_d).$$

Therefore Γ^I is contained in the center of Γ and Γ/Γ^I is cyclic. So Γ must be abelian. □

Corollary 3.3.6. *There exists a character $\chi : G_K \to E^\times$ such that $V_1 \otimes \chi$ and $V_2 \otimes \chi^{-1}$ are semi-stable.*

Proof. Notations as in the above lemma. Since Γ is abelian, there exists a totally ramified abelian extension K_1 and a unramified extension K_2 such that $K_1 K_2 = K'$. In particular, $\mathrm{Gal}(K_1/K) \simeq \Gamma^I$. The above lemma shows that Γ^I acts on $D_{\mathrm{st}}(V_1)$ via a character $c : \Gamma^I \to E^\times$. So the isomorphism $\mathrm{Gal}(K_1/K) \simeq \Gamma^I$ induces a character $\tilde{\chi} : \mathrm{Gal}(K_1/K) \to E^\times$. It is easy to check that $\chi = \tilde{\chi}^{-1}$ is just what we want. □

Now we return to the Question 3.3.1 of when such a character χ exists. Note that one can easily formulate the analogue of Question 3.3.1 for local Galois representations. The answer to this analogy is indeed affirmative which is proved by Di Matteo [DM13].

Theorem 3.3.7 ([DM13])**.** *Let K be a finite extension of $\mathbb{Q}_p$ and denote $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. Assume that $V_i : G_K \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$, $i = 0, 1$, are two Galois representations such that $V_0 \otimes V_1$ are semi-stable (resp. crystalline). Then there exists a character $\chi : G_K \to \overline{\mathbb{Q}}_p^\times$ such that $V_0 \otimes \chi$ and $V_1 \otimes \chi^{-1}$ are semi-stable (resp. crystalline).*

Just as Question 3.3.1 and Question 3.3.2, the above theorem can be formulated in more general settings and has been proved in [Con] and [Pat13].

While one can directly to use Di Matteo's theorem give a quick proof of Proposition 3.3.4, our proof of the proposition gives another proof to Di Matteo's theorem (but the essential difference is only the proof of Theorem A.0.1 in the appendix), provided there exists a character χ so that $V_0 \otimes \chi$ and $V_1 \otimes \chi^{-1}$ are Hodge-Tate. And the proof of the existence of such a χ will be contained in the proof of Theorem 3.3.9.

Since any character of $G_{\mathbb{Q}_p}$ can be extended to a character of $G_{\mathbb{Q}}$. We obtain the following result:

Corollary 3.3.8. *If $F = \mathbb{Q}$ then the answer to Question 3.3.1 is affirmative.*

If $F \neq \mathbb{Q}$ then the situation is much more complicated.

For the rest of this subsection, we specialize to the situation mentioned at the beginning of the section: $\rho_F := \rho|_{G_F} = \rho_0 \otimes \rho_1$ with $\rho : G_{\mathbb{Q}} \to \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$.

Let ϵ_p denote the p-adic cyclotomic character. After replacing ρ by $\rho \otimes \epsilon_p^k$ for some integer k, we may assume that the Hodge-Tate weights of ρ are of the form $0, m, n, l$ with $l \geq m, n \geq 0$. It is easy to see by the self-duality that $l = m + n$.

Theorem 3.3.9. *Assumption as the above. If $m \equiv n \mod 2$, then the answer to Question 3.3.1 is affirmative.*

To proceed with the proof, we modify the idea of Di Matteo to deal with the Hodge-Tate weights of global representations. We first briefly recall Sen's operator Θ defined in [Sen81][1]. Let K be a finite extension of $\mathbb{Q}_p$, ζ_{p^n} a primitive p^n-th root of unity, $K_\infty := \bigcup_{n\geq 1} K(\zeta_{p^n})$, $H := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K_\infty)$ and $\Gamma := \mathrm{Gal}(K_\infty/K)$. Let W be a d-dimensional $\mathbb{C}_p$-vector space with a continuous $\mathbb{C}_p$-semi-linear action of $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. Then one can show that $\hat{D}(W) := W^H$ is a finite $\hat{K}_\infty$-vector space of dimension d, where $\hat{K}_\infty$ is the closure of K_∞ in $\mathbb{C}_p$. There exists a unique K_∞-subspace of $D(W)$ such that $D(W) \otimes_{K_\infty} \hat{K}_\infty = \hat{D}(W)$ and Γ stabilizes $D(W)$ (see Theorem 3 in [Sen81]). From the construction of $D(W)$, one can easily prove that $D(W_1 \oplus W_2) = D(W_1) \oplus D(W_2)$ and $D(W_1 \otimes_{\mathbb{C}_p} W_2) = D(W_1) \otimes_{K_\infty} D(W_2)$. By Theorem 4 in [Sen81], there exists a K_∞-linear operator $\Theta_{D(W)}$ on $D(W)$ such that for any $w \in D(W)$ there exists an open subgroup $\Gamma_w \subset \Gamma$ such that

$$\sigma(w) = [\exp(\Theta_{D(W)} \log(\epsilon_p(\sigma)))](w), \text{ for any } \sigma \in \Gamma_w.$$

If V is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous $\mathbb{Q}_p$-linear G_K-action, then we consider the operator $\Theta_{D(V_{\mathbb{C}_p})}$ on $D(V_{\mathbb{C}_p})$ where $V_{\mathbb{C}_p} := \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. It turns out that Θ enjoys the following properties:

Proposition 3.3.10 ([Sen81])**.** (1) *There exists a basis in $D(W)$ such that the coefficients of the matrix of $\Theta_{D(W)}$ are in K.*

[1] Sen use φ to denote Θ in the original paper.

(2) *V is Hodge-Tate if and only if $\Theta_{D(V_{\mathbb{C}_p})}$ on $D(V_{\mathbb{C}_p})$ is semi-simple with eigenvalues in $\mathbb{Z}$.*

(3) $\Theta_{D(W_1 \otimes_{\mathbb{C}_p} W_2)} = \Theta_{D(W_1)} \otimes 1 + 1 \otimes \Theta_{D(W_2)}$.

In particular, if Θ on $D(W_1 \otimes_{\mathbb{C}_p} W_2)$ is semi-simple then Θ on $D(W_1)$ and $D(W_2)$ are semi-simple. Let $E \subset \mathbb{C}_p$ be a finite Galois extension over $\mathbb{Q}_p$ such that $K \subset E$. Set $J' = \mathrm{Gal}(E/\mathbb{Q}_p)$. Let V be a finite dimensional E-vector space with a continuous E-linear G_K-action. Note that $V_{\mathbb{C}_p} = V \otimes_{\mathbb{Q}_p} \mathbb{C}_p = \bigoplus_{\sigma \in J'} V \otimes_{E,\sigma} \mathbb{C}_p$. Write $V_{\mathbb{C}_p,\sigma} := V \otimes_{E,\sigma} \mathbb{C}_p$. We see that $V_{\mathbb{C}_p,\sigma}$ has a semi-linear $G_E := \mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$-action. So one can still consider Θ on $D(V_{\mathbb{C}_p,\sigma})$. For each $\tau \in \mathrm{Gal}(E/K)$, one can check that τ induces an isomorphism between $D(V_{\mathbb{C}_p,\sigma})$ to $D(V_{\mathbb{C}_p,\tau\sigma})$ and the isomorphism commutes with Θ on $D(V_{\mathbb{C}_p,\sigma})$ and $D(V_{\mathbb{C}_p,\tau\sigma})$. Therefore, if we write $\mathrm{HT}_\sigma(V)$ for the set of eigenvalues of Θ on $D(V_{\mathbb{C}_p,\sigma})$ then $\mathrm{HT}_\sigma(V)$ only depends on the set of cosets $J'/\mathrm{Gal}(E/K)$, which is also the set J of all embeddings $\sigma : K \to \overline{\mathbb{Q}}_p$.

Let L be a number field and V a finite dimensional E-vector space with a continuous E-linear G_L-action. Assume that E contains all embeddings of L to $\overline{\mathbb{Q}}_p$. For each prime $v|p$ and $\tau \in \mathrm{Gal}(E/\mathbb{Q}_p)$, we can consider Θ on $D(V_{\mathbb{C}_p,\tau})$ restricted to G_{L_v}. One easily prove that Θ only depends on the embeddings $\sigma : L \to \overline{\mathbb{Q}}_p$ as the above. We write J for the set of all embeddings $\sigma : L \to \overline{\mathbb{Q}}_p$ and $\mathrm{HT}_\sigma(V)$ the set of eigenvalues of Θ for each $\sigma \in J$.

Now consider that U and U' are finite dimensional E-vector spaces with continuous E-linear G_K-actions such that $V := U \otimes_E U'$ is Hodge-Tate. Note that

$$V_{\mathbb{C}_p} = (U \otimes_E U') \otimes_{\mathbb{Q}_p} \mathbb{C}_p = \bigoplus_{\sigma \in J'} U \otimes_E (U' \otimes_{E,\sigma} \mathbb{C}_p) = \bigoplus_{\sigma \in J'} (U \otimes_{E,\sigma} \mathbb{C}_p) \otimes_{\mathbb{C}_p} (U' \otimes_{E,\sigma} \mathbb{C}_p).$$

So Θ on $D(U_{\mathbb{C}_p,\sigma})$ and $D(U'_{\mathbb{C}_p,\sigma})$ is semisimple and if $\mathrm{HT}_\sigma(U) = \{s_1^\sigma, \dots, s_a^\sigma\}$, $\mathrm{HT}_\sigma(U') = \{t_1^\sigma, \dots, t_b^\sigma\}$ then $\mathrm{HT}_\sigma(V) = \{s_i^\sigma + t_j^\sigma,\ i = 1, \dots, a, j = 1, \dots, b\}$. Hence $s_i^\sigma + t_j^\sigma \in \mathbb{Z}$ as V is Hodge-Tate. Then $s_i^\sigma - s_1^\sigma \in \mathbb{Z}$ for $i = 1, \dots a$ and $bs_1^\sigma + \sum_{j=1}^b t_j^\sigma \in \mathbb{Z}$. By Proposition 3.3.10 (1), we see that $s_1^\sigma \in K$. Let L be a number field and J the set of all embeddings of L to $\overline{\mathbb{Q}}_p$. Then the above statement is still valid for each $\sigma \in J$ if U and U' are finite dimensional E-vector spaces with continuous E-linear G_L-action such that $V := U \otimes_E U'$ are Hodge-Tate at each prime $v|p$.

Suppose that U and U' are E-representations of G_K with K a finite extension of $\mathbb{Q}_p$. For any $x \in K$, there always exists a character $\chi_\sigma : G_K \to {E'}^\times$ for a finite extension E' over K such that $\mathrm{HT}_\sigma(\chi_\sigma) = \{x\}$ and $\mathrm{HT}_\tau(\chi_\sigma) = \{0\}$ for any $\tau \neq \sigma$ (see Lemma 2.1.3 in [DM13]). Hence by enlarging E if necessary, there exists a character χ such that $U \otimes_E \chi$ and $U' \otimes_E \chi^{-1}$ are Hodge-Tate [2].

If U and U' are E-representations of G_L with L a number field. Then the existence of the above character χ is much more complicated (unless $L = \mathbb{Q}$ as a character of $G_{\mathbb{Q}_p}$ can be always extended to a character of $G_{\mathbb{Q}}$, c.f. Corollary 3.3.8). Now let us return the situation of ρ_0 and ρ_1 in Theorem 3.3.9. Let J be the set of embeddings of F to $\overline{\mathbb{Q}}_p$. Now assume that $\mathrm{HT}_\sigma(\rho_0) = \{a_\sigma, a_\sigma + s_\sigma\}$ and $\mathrm{HT}_\sigma(\rho_1) = \{b_\sigma, b_\sigma + t_\sigma\}$ for each $\sigma \in J$. As the discussion above, we can assume that s_σ, t_σ are integers.

[2] This is the exactly missing ingredient if we want to reprove Theorem 3.3.7.

Lemma 3.3.11. *There exists a character $\chi : G_F \to \overline{\mathbb{Q}}_p^\times$ such that for each $\sigma \in J$, we have $\mathrm{HT}_\sigma(\chi) = \{a_\sigma + \frac{s_\sigma}{2}\}$.*

Proof. Let $\chi'' := \det(\rho_1)$. Then we see that $\mathrm{HT}_\sigma(\chi'') = \{2a_\sigma + s_\sigma\}$. Select E big enough such that $\chi''(G_F) \subset \mathcal{O}_E^\times$. Modulo the group of torsion points of $\mathcal{O}_E^\times$, we get a character χ' such that $\chi'(G_F) \subset 1 + \pi\mathcal{O}_E$ with π a uniformizer of E. If $p \neq 2$ then $\chi := (\chi')^{1/2}$ is the required character. For $p = 2$, there exists finite Galois extension F'/F so that $(\chi')^{1/2}$ makes sense (when $\chi'(G_{F'}) \subset 1+4\mathcal{O}_E$). It is elementary to extend this square root to a character $\chi : G_F \to \mathcal{O}_{E'}^\times$ for some finite extension E'/E. □

Now after twisting χ^{-1} to ρ_0, $\mathrm{HT}_\sigma(\rho_0) = \{-\frac{s_\sigma}{2}, \frac{s_\sigma}{2}\}$ for each $\sigma \in J$. As the above lemma, there exists a character $\alpha : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ such that $\mathrm{HT}(\alpha) = \frac{1}{2}$. Now fix a $\tau \in J$ and twisting ρ_0 by α^{s_τ}, we get $\mathrm{HT}_\sigma(\rho_0) = \{\frac{s_\sigma - s_\tau}{2}, \frac{s_\sigma + s_\tau}{2}\}$. Now as $\rho_0 \otimes \rho_1$ has weight $0, m, n$, and $m+n$, we can assume that $s_\tau = m$ and $\mathrm{HT}_\tau(\rho_1) = \{0, n = t_\tau\}$. So $\mathrm{HT}_\sigma(\rho_1) = \{-\frac{s_\tau - s_\sigma}{2}, m + n - \frac{s_\tau + s_\sigma}{2}\}$. Now there are two possibilities for s_σ: Either $s_\sigma = m$ or $s_\sigma = n$. By the hypothesis $m \equiv n \mod 2$, we see both $\mathrm{HT}_\sigma(\rho_i)$ are set of integers for any $\sigma \in J$ and $i = 0, 1$. Hence both ρ_i are Hodge-Tate. By Proposition 3.3.4, this completes the proof of Theorem 3.3.9.

If $m \not\equiv n \mod 2$ then the answer to Question 3.3.1 is not always positive. See examples constructed from mixed-parity Hilbert modular forms in §8 and §9 of [Pat13].

3.4. **Irreducibility.** In this subsection, we again assume that $\rho_F := \rho|_{G_F} = \rho_0 \otimes \rho_1$ comes from $\rho : G_{\mathbb{Q}} \to \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$. We obviously have

Lemma 3.4.1. *If ρ_F is irreducible, then so are ρ_0 and ρ_1.*

Next, we will assume that ρ is irreducible as a 4-dimensional representation and consider the irreducibility of ρ_F. This is only interesting when $F \neq \mathbb{Q}$, which we assume from now on. We will further assume that $\rho \not\simeq \mathrm{ind}_{G_L}^{G_{\mathbb{Q}}} \chi$ for any character χ of G_L such that $[L : \mathbb{Q}] = 4$ and L contains a quadratic extension of $\mathbb{Q}$ (for if $\rho \simeq \mathrm{ind}_{G_L}^{G_{\mathbb{Q}}} \chi$, the automorphy of ρ can be easily proved by automorphic induction in [AC89]). Finally we assume that $\rho_p := \rho|_{G_{\mathbb{Q}_p}}$ is Hodge-Tate and $\mathrm{HT}(\rho) = \{0, m, n, m+n\}$ with $m + n > 0$.

Proposition 3.4.2. *With the above assumptions, $\rho|_{G_M}$ is irreducible for any real quadratic field M.*

Proof. Denote by W the representation $\rho|_{G_M}$ and assume that W is not irreducible. By Clifford theory in [Cli37], W is the direct sum of two irreducible subrepresentations V_1, V_2 and $V_2 = V_1^\tau$, where τ is the non-trivial element of $\mathrm{Gal}(M/\mathbb{Q})$. Obviously, for $i = 1, 2$, the image of $G_M \to \mathrm{GL}(V_i)$ lies in $\mathrm{GO}(V_i, Q|_{V_i})$. We claim that $Q|_{V_i}$ is either 0 or non-degenerate. Otherwise, the kernel of $Q|_{V_i}$ is a 1-dimensional subspace invariant under G_M, contradicting irreducibility.

Consider the case $Q|_{V_i}$ is non-degenerate. It is well-known that this implies that V_i is induced from a character of a subgroup G_L of index 2 in G_M, contradicting our assumptions.

Therefore, we must have $Q|_{V_i} = 0$ for both $i = 1, 2$. This implies $V_2 = V_1^* \otimes (r \circ \rho)$ as G_M-modules, where V_1^* denotes the dual of V_1. Hence $V_2 = V_1 \otimes \chi$ with $\chi = (r \circ \rho) \det(V_1)^{-1}$.

Let $V = V_1$ and $V^\tau = V_2$. Let us discuss the Hodge-Tate weights of V and V^τ. Let $J := \{\sigma, \sigma' : M \to \overline{\mathbb{Q}}_p\}$ be the set of all embeddings from M to $\overline{\mathbb{Q}}_p$. For each $\alpha \in J$, we have

$$\mathrm{HT}_\alpha(V) \cup \mathrm{HT}_\alpha(V^\tau) = \mathrm{HT}(\rho) = \{0, m, n, m+n\},$$

and $\mathrm{HT}_\sigma(V) = \mathrm{HT}_{\sigma'}(V^\tau)$. By Lemma 3.4.3, we get $\mathrm{HT}_\sigma(\det(V)) = \mathrm{HT}_{\sigma'}(\det(V))$ and $\mathrm{HT}_\sigma(\det(V^\tau)) = \mathrm{HT}_{\sigma'}(\det(V^\tau))$. These conditions force that $\mathrm{HT}_\sigma(V) = \{0, m+n\}$, $\mathrm{HT}_{\sigma'}(V) = \{m, n\}$, $\mathrm{HT}_\sigma(V^\tau) = \{m, n\}$ and $\mathrm{HT}_{\sigma'}(V^\tau) = \{0, m+n\}$. Note that $V^\tau \simeq V \otimes \chi$ implies that either $m = 0$ or $n = 0$ (note that χ has only one weight), and the weight of χ has to be 0. So χ is a finite character. Then there exists a finite extension L of M, $V \simeq V^\tau$ when restricted to L. We have two situations here. Case 1: V restricted to L is reducible; Case 2: V restricted to L is irreducible.

Let us first deal with the first case. We first claim that L can be chosen to be a quadratic extension of M. If so then ρ is induced from a character of G_L with $[L : \mathbb{Q}] = 4$. To see the claim, write $V|_{G_L} = U \oplus U'$. Note that U and U' can not be isomorphic as $\mathrm{HT}_\sigma(V) = \{0, m+n\}$ and $m + n > 0$ for any embedding $\sigma : L \to \overline{\mathbb{Q}}_p$. Set $H := \{g \in G_M | U^g = U\}$. Since V is irreducible and U and U' are not isomorphic, H is an index 2-subgroup of G_M. Then the fixed field of H is just what we want.

Now consider the case that V is irreducible over L. Note in this case, we have all irreducible components of ρ are isomorphic. By Theorem 3 in [Cli37] and Theorem 2.2.1, there exist two representations W_1 and W_2 of $G_\mathbb{Q}$ such that $W_1 \otimes W_2 \simeq \rho$. Hence $\rho(G_\mathbb{Q}) \subset \mathrm{SGO}_4(\overline{\mathbb{Q}}_p)$. But this contradicts the hypothesis that $F \neq \mathbb{Q}$. Finally, we have treated all the possibilities and proven Proposition 3.4.2. $\square$

Lemma 3.4.3. *Let χ be a Hodge-Tate character of $G_M \to \overline{\mathbb{Q}}_p^\times$ with M a totally real field. Then $\mathrm{HT}_\sigma(\chi)$ is a unique integer independent on σ.*

Proof. See the discussion above Lemma 4.1.3 in [CHT08] or [Ser89]. $\square$

4. Proof of the main results

4.1. **Definition of Automorphy.** Let L be a number field and $\rho : G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ a continuous representation. We call ρ *automorphic* if there exists an automorphic representation $\pi \simeq \otimes'_v \pi_v$ of $\mathrm{GL}_n(\mathbb{A}_L)$ such that for almost all primes v the (Frobenius-semi-simplification of the) Weil-Deligne representation of $\rho|_{G_{L_v}}$ is isomorphic to the Weil-Deligne representation associated to π_v via the local Langlands correspondence. In particular, we call ρ *modular* if π is obtained from a modular form. By definition, ρ is called *potentially automorphic* if there exists a finite extension L' of L such that $\rho|_{G_{L'}}$ is automorphic.

Let $r_i : G_L \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ for $i = 0, 1$ be continuous representations. By the main theorem of [Ram00], if both r_i are automorphic then $r_1 \otimes r_2$ is automorphic. So in the following, we use potential automorphy theorems of GL_2 to prove the theorems in §1.

4.2. **Potential automorphy theorem of** GL_2**.** We first summarize the known (potential) automorphy theorems of GL_2 from [BLGGT11], [DFG04] and [Die08].

Theorem 4.2.1. *Let F be a totally real field and $\sigma : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ a continuous representation. Assume the following:*

1. *σ is irreducible and unramified almost everywhere;*
2. *σ is totally odd, i.e., for each complex conjugation c, $\det(\rho)(c) = -1$;*
3. *F is unramifed at p; for each prime $v|p$, $\sigma|_{G_{F_v}}$ is crystalline and for each embedding $\tau : F \to \overline{\mathbb{Q}}_p$, $\mathrm{HT}_\tau(\sigma) = \{a_\tau, a_\tau + b_\tau\}$ with $0 < b_\tau < p - 1$;*
4. *$\bar{\sigma}|_{G_{F(\zeta_p)}}$ is irreducible, where $\bar{\sigma}$ denotes the reduction of σ;*
5. *$p \geq 6$.*

Then there exists a finite Galois totally real extension F'/F such that $\sigma|_{G_{F'}}$ is automorphic.

If $F = \mathbb{Q}$ we only need to assume (1), (2), (3), (4) and then σ is modular. If $F = \mathbb{Q}$ and we assume (1), (2), (3) and $p + 1 \nmid 2b_\tau$ then σ is modular.

Proof. The first part of the theorem is the special case of Theorem C in [BLGGT11] for $n = 2$. If $F = \mathbb{Q}$ and we assume (1), (2), (3) and (4) then the main result in [DFG04] implies that σ is modular with the input of Serre's conjecture. Finally if we assume that (1), (2), (3) and that $p + 1 \nmid 2b_\tau$ then [Die08] proved that σ is modular. □

Remark 4.2.2. If $F = \mathbb{Q}$ then by the recent work of Calegari, Emerton and Kisin, conditions (2) and (3) can be relaxed significantly. See [Cal12], [Eme11] and [Kis09] for more details. Unfortunately, when relaxing the conditions (2) (3), they need to impose some conditions on residual representations, which is not easy to check in applications. So here we select an easy version of automorphy theorem without hypothesis for residual representations. We remark that (2) (3) always holds for enough big prime p if we consider the regular compatible systems coming from geometry.

Proof of Theorem 1.0.1 and Theorem 1.0.5. Now using the above theorem and the main theorem in [Ram00], combined with the discussion in §3, we prove Theorem 1.0.1 and Theorem 1.0.5. □

Remark 4.2.3. There is another way to prove potentially automorphy for ρ pointed out by Calegari: Since $\rho|_{G_F} \simeq \rho_0 \otimes \rho_1$, we have

$$\bigwedge^2 \rho|_{G_F} \simeq \mathrm{sym}^2(\rho_0)\det(\rho_1) \oplus \mathrm{sym}^2(\rho_1)\det(\rho_0).$$

Hence $\mathrm{sym}^2(\rho_0)\det(\rho_1)$ and $\mathrm{sym}^2(\rho_1)\det(\rho_0)$ are crystalline. After twisting by a character, we can assume that $\det(\rho_i)$ are Hodge-Tate, hence potentially crystalline at each $v|p$. Then one can apply potential automorphy theorem for GO_3 from [BLGGT11] to $\mathrm{sym}^2(\rho_0)\det(\rho_1)$ and $\mathrm{sym}^2(\rho_1)\det(\rho_0)$, and then prove potential automorphy of $\mathrm{sym}^2(\rho_0)\det(\rho_1)$ and $\mathrm{sym}^2(\rho_1)\det(\rho_0)$. Hence both $\mathrm{sym}^2(\rho_i)$ are potentially automorphic. Finally, by [Ram], we conclude that ρ_i are potentially automorphic and then ρ are potentially automorphic. This strategy skips the steps showing that ρ_i are crystalline (one still needs to show that the $\det(\rho_i)$ are Hodge-Tate after twisting by a character) and does not need the restriction that $m \equiv n \mod 2$. On the other hand, this strategy cannot prove automorphy of ρ even assuming that $\rho(G_{\mathbb{Q}}) \subset \mathrm{SGO}_4(\overline{\mathbb{Q}}_p)$, and one has to impose stronger conditions on Hodge-Tate weights and residual representations in order to use the potential automorphy theorem of GO_3.

Now let us discuss the situation of compatible system of Galois representations. Let L be a number field. For each prime v of $\mathcal{O}_L$, we use $\mathrm{rch}(v)$ to denote the residue characteristic of v. Following [BLGGT11], we define *a rank n compatible system of p-adic Galois representations $\mathcal{R}$ of G_L defined over E* to be a 5-tuple

$$\{E, S, \{Q_l[X]\}, \{\rho_\lambda\}, \{\mathbf{v}_\tau\}\}$$

where

- E is a number field;
- S is a finite set of primes of L;
- for each prime $l \notin S$ of L, $Q_l[X]$ is a monic degree n polynomial in $E[X]$;
- for each prime λ of E with $\mathrm{rch}(\lambda) = p$

  $$\rho_\lambda : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(E_\lambda)$$

  is a continuous, semi-simple, representation such that
  (1) if $l \notin S$ and rch $(l) \neq p$ then ρ_λ is unramified at l and $\rho_\lambda(\mathrm{Frob}_l)$ has characteristic polynomial $Q_l(X)$;
  (2) if $l|p$ then $\rho_\lambda|_{G_{L_l}}$ is de Rham and in the case $l \notin S$ crystalline;
- for $\tau : L \to \overline{\mathbb{Q}}_p$, $\mathbf{v}_\tau$ is a fixed multi-set of integers such that $\mathrm{HT}_\tau(\rho_\lambda) = \mathbf{v}_\tau$.

If $L = \mathbb{Q}$ then we simply drop the trivial embedding τ from subscripts of $\mathbf{v}$ and HT. Note that our definition of compatible system is *weakly* compatible system in the sense of [BLGGT11, §5.1] with one slightly difference: We require that $\rho_\lambda(G_{\mathbb{Q}}) \subset \mathrm{GL}_d(E_\lambda)$ instead of $\rho_\lambda(G_{\mathbb{Q}}) \subset \mathrm{GL}_d(\overline{E}_\lambda)$ defined in §5.1 in [BLGGT11]. Since we only concern about the representations of GO_4-type or GL_2-type, we further assume that

- $n = 4$ or $n = 2$, and ρ_λ is absolutely irreducible for each λ;
- If $n = 4$ then $\rho_\lambda(G_{\mathbb{Q}}) \subset \mathrm{GO}_4(\overline{\mathbb{Q}}_p)$ for each λ;
- If $n = 4$ then the multiplier $\{r(\rho_\lambda)\}$ also forms a compatible system;
- If $n = 4$ then the eigenvalues of complex conjugation c on ρ_λ are $-1, -1, 1, 1$ and $r(\rho_\lambda(c)) = 1$ for some λ (hence for all λ).

Now we are ready to prove Theorem 1.0.6. By Theorem 4.2.1 and the definition of compatible system, we only need to show that there exists a prime p large enough such that $\bar{\rho}_\lambda|_{G_{F(\zeta_p)}}$ is irreducible. For this, we modify the proof of [BLGGT11, Prop. 5.3.2] to the following lemma to deal with ρ_λ which is not *regular*. Recall that regularity means that $\mathbf{v}_\tau = \mathrm{HT}_\tau(\rho_\lambda)$ consists of distinct integers. Note that [BLGGT11, Prop. 5.3.2] only treats regular compatible systems.

Lemma 4.2.4. *Suppose that $\{E, S, \{Q_l[X]\}, \{r_\lambda\}, \{\mathbf{v}_\tau\}\}$ is a rank 4 compatible system of G_F with $\mathbf{v}_\tau = \{0, m, m, 2m\}$ and $m > 1$. Assume that r_λ is absolutely irreducible for each λ. Then there is a set of rational primes L of Dirichlet density 1 such that if $p = \mathrm{rch}(\lambda) \in L$ then $\bar{r}_\lambda|_{G_{F(\zeta_p)}}$ is absolutely irreducible .*

Proof. The proof of [BLGGT11, Prop. 5.3.2] still works if we can reproduce Lemma 5.3.1 (*loc. cit.*) in our situation. First Lemma 5.3.1 (1) (which is a result of Serre) is always valid without the assumption of regularity. So it suffices to reprove Lemma 5.3.1 (2) (3), that is,

(1) If H is an open subgroup of G_F then any irreducible H-subrepresentation s of $\overline{\mathbb{Q}}_p \otimes_{E_\lambda} r_\lambda$ has multiplicity one.
(2) After replacing E by a finite extension, we may assume that for any open subgroup $H \subset G_F$ and any λ and any H-subrepresentation s of $\overline{\mathbb{Q}}_p \otimes_{E_\lambda} r_\lambda$, the representation s is defined over $\mathcal{O}_{E_\lambda}$.

Since r_λ has been assumed to be absolutely irreducible, Theorem 2 in [Cli37] implies that any other irreducible H-subrepresentation of r_λ is s^γ with $\gamma \in G_F$. In particular, there exists a $\gamma_0 \in G_F$ such that $0 \in \mathrm{HT}_\sigma(s^{\gamma_0})$. Hence s^{γ_0} must has multiplicity one because 0 has multiplicity one in $\mathrm{HT}_\tau(r_\lambda)$. Therefore, s has multiplicity one. To show that s is defined over $\mathcal{O}_{E_\lambda}$, we may assume that $0 \in \mathrm{HT}_\sigma(s)$ as the above argument. For any $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/E_\lambda)$, write $g(s) = \overline{\mathbb{Q}}_p \otimes_{g,\overline{\mathbb{Q}}_p} s$. Since r_λ has been assumed to be defined over E_λ (note that here our assumption is slightly different from that in [BLGGT11], where they only assume that characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_l)$ is defined over E, and then showed (by using regularity) that r_λ is defined over E_λ after replacing E by a finite extension), $g(s)$ is a subrepresentation of $\overline{\mathbb{Q}}_p \otimes_{E_\lambda} r_\lambda$. It suffices to show that $g(s) = s$ (Here we thank Richard Taylor for teaching us this trick). By Lemma 5.3.1 (1) of [BLGGT11] (this part of Lemma is due to Serre), there exists a finite Galois number field F^1 so that if s is an irreducible G_{F_1}-subrepresentation of $\overline{\mathbb{Q}}_p \otimes_{E_\lambda} r_\lambda$ then $s|_{G_{F'}}$ remains irreducible for any finite extension F'/F^1. So without loss of generality and enlarging E, we may assume that $F^1 \subset E$ and $G_{F^1} \subset H$. By the discussion after Proposition 3.3.10 of $\mathrm{HT}_\sigma(V)$ and note that $F^1 \subset E$, we see that $\mathrm{HT}_\sigma(g(s)) = \mathrm{HT}_\sigma(s)$ (this trick has been used in [Pat13]). Then this forces $g(s) = s$ as $0 \in \mathrm{HT}_\tau(r_\lambda)$ has multiplicity one. Therefore s is defined over E_λ. □

Now since ρ_λ are absolutely irreducible by assumption, Proposition 3.4.2 shows that $\rho_\lambda|_{G_F}$ are absolutely irreducible (unless ρ_λ is an induction of a character). Then the above lemma shows that there exists a prime p large enough such that $\bar\rho_\lambda|_{G_{F(\zeta_p)}}$ is irreducible. This completes the proof of Theorem 1.0.6.

4.3. **The special case when $\mathbf{v} = \{0, 0, m, m\}$.** In this subsection, we consider a compatible system $\mathcal{R}$ satisfying the following extra conditions:

- $\rho_\lambda(G_\mathbb{Q}) \subset \mathrm{SGO}_4(\overline{\mathbb{Q}}_p)$ for each λ;
- Eigenvalues of c on ρ_λ are $-1, -1, 1, 1$ and $r(\rho_\lambda)$ forms a compatible system with $r(\rho_\lambda(c)) = 1$ for a λ (hence for all λ);
- $\mathbf{v} = \{0, 0, m, m\}$ with $m > 0$.

We want to show that the above system is automorphic. So far we have shown that for each λ, there exist $\rho_{i,\lambda} : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ for $i = 0, 1$ such that $\overline{\mathbb{Q}}_p \otimes_{E_\lambda} \rho_\lambda \simeq \rho_{0,\lambda} \otimes \rho_{1,\lambda}$, $\rho_{i,\lambda}|_{G_{\mathbb{Q}_p}}$ are crystalline if $\rho_\lambda|_{G_{\mathbb{Q}_p}}$ are crystalline and $\mathrm{HT}(\rho_{0,\lambda}) = \{0, 0\}$, $\mathrm{HT}(\rho_{1,\lambda}) = \{0, m\}$ with $m > 0$. It is not hard to see that we can arrange $\rho_{i,\lambda}$ so that $\rho_{i,\lambda}$ are unramifed over $l \notin S \cup \{\mathrm{rch}(\lambda)\}$ (here we crucially use the fact that the base field is $\mathbb{Q}$). We have already shown that $\rho_{1,\lambda}$ is modular by Theorem 4.2.1 if $\mathrm{rch}(\lambda)$ is big enough. If $\{\rho_{0,\lambda}\}$ forms a weakly compatible system then we can use Kisin and Serre's strategy in the proof of Theorem (1.3.1) (Artin conjecture) in [Kis07] to show $\{\rho_{0,\lambda}\}$ is modular. Unfortunately it is not clear that $\{\rho_{0,\lambda}\}$ forms a weakly compatible system though $\{\rho_\lambda\}$ forms a weakly compatible system. Fortunately, we can still modify Kisin and Serre's method to prove that ρ_λ is modular by 3 steps.

First step: we show there exists a positive integer N and a set Σ of infinitely many primes $\lambda \in \mathrm{Spec}(\mathcal{O}_E)$ such that

$$\mathrm{cond}(\rho_{0,\lambda}), \mathrm{cond}(\rho_{1,\lambda}) | N$$

where $\mathrm{cond}(V)$ denotes the swan conductor of a Galois representation of V. It turns out this is the most technical part, which we prove in the end of this subsection.

Second step: We denote $\{\rho_{f,\lambda}\}$ the weakly compatible system of 2-dimensional Galois representations of $G_{\mathbb{Q}}$ associated to the modular form f. Let V (resp. V') be a finite dimensional L (resp. L')-vector space with continuous L (resp. L')-linear $G_{\mathbb{Q}}$-action. We write $V \sim V'$ if L and L' have the same algebraic closure $\overline{L}$ and $\overline{L} \otimes_L V \simeq \overline{L} \otimes_{L'} V'$ as $G_{\mathbb{Q}}$-representations.

Note that $\rho_{1,\lambda} \sim \rho_{f_\lambda,\lambda}$ for a modular form f_λ if $p = \mathrm{rch}(\lambda)$ is large enough (more precisely, if $p > 2m$ and $p \notin S$). Without loss of generality, we may assume that if $\lambda \in \Sigma$ then $\rho_{1,\lambda}$ is modular. By the first step all modular forms corresponding to $\rho_{1,\lambda}$ are in the space $S_{m+1}(N, \mathbb{C})$, which is the space of cusp forms with level N and weight $m+1$. Hence there are only finitely many normalized eigenforms. So there are infinitely many λ such that $\rho_{1,\lambda}$ attaches to one eigenform f. Without loss of generality, we may assume that $\rho_{1,\lambda}$ comes from one eigenform f for all $\lambda \in \Sigma$. Note that there exists a number field E_f such that for any $\lambda \in \Sigma$ there exists a prime $\lambda' \in \mathrm{Spec}(\mathcal{O}_{E_f})$ with the same residue characteristic such that $\rho_{1,\lambda} \simeq \overline{\mathbb{Q}}_p \otimes_{E_{f,\lambda'}} \rho_{f,\lambda'}$. So by enlarging E, we may assume that $\rho_{1,\lambda}(G_{\mathbb{Q}}) \subset \mathrm{GL}_2(E_\lambda)$. It is not clear that $\rho_{0,\lambda}(G_{\mathbb{Q}}) \subset \mathrm{GL}_2(E_\lambda)$ (here we thank for the referee pointing this out). Luckily we do not need this in the following proof.

Last step: Now we can follow the similar idea of Kisin and Serre (see, for example, the proof of Theorem (1.2.1) and Theorem (1.3.1) in [Kis07]). First, we prove there exists infinitely many primes $\lambda \in \Sigma' \subset \Sigma$ such that the residue representation of $\rho_{0,\lambda}$ is absolutely irreducible. The proof is almost the same as the last part of the proof in Theorem (1.2.1) in [Kis07]. Let us sketch the proof here. Let $\bar{V}$ denote the *semi-simplification* of the residual representation of Galois representation V. Assume that there are infinitely many primes $\lambda \in \Sigma$ such that $\bar{\rho}_{0,\lambda}$ are reducible. Then $\bar{\rho}_{0,\lambda} = \epsilon_{1,\lambda} \oplus \epsilon_{2,\lambda}$ with $\epsilon_{i,\lambda} : G_{\mathbb{Q}} \to \overline{\mathbb{F}}^\times$ characters. We lift $\epsilon_{i,\lambda}$ to characters $\hat{\epsilon}_{i,\lambda} : G_{\mathbb{Q}} \to \overline{\mathbb{Z}}^\times$ for $i = 1, 2$. Since the conductors of $\rho_{0,\lambda}$ are bounded, we see that the conductors of $\hat{\epsilon}_{i,\lambda}$ are bounded (note that $\epsilon_{i,\lambda}$ are unramified at p). Therefore there are only finitely many such characters. Without loss of generality, we may assume that $\hat{\epsilon}_{i,\lambda} = \chi_i$ for each $\lambda \in \Sigma''$ and $i = 1, 2$, where $\Sigma'' \subset \Sigma$ is a subset of infinitely many primes and $\chi_i : G_{\mathbb{Q}} \to \overline{\mathbb{Z}}^\times$ for $i = 1, 2$ are Dirichlet characters. Enlarge E such that E is Galois and contains $\chi_i(G_{\mathbb{Q}})$ for $i = 1, 2$. Let $\mathfrak{m}_\lambda$ denote the maximal ideal of $\mathcal{O}_{\overline{E}_\lambda}$. Then there exist infinitely many primes λ of $\mathcal{O}_E$ such that $\mathrm{tr}(\chi_1 \oplus \chi_2) \equiv \mathrm{tr}(\rho_{0,\lambda}) \mod \mathfrak{m}_\lambda$ where tr stands for trace. In particular, for a fixed such λ_0 and any fixed rational prime $l \neq \mathrm{rch}(\lambda_0)$ such that $l \notin S$, by the compatibility of ρ_λ, we get

$$\begin{aligned}
\mathrm{tr}(((\chi_1 \oplus \chi_2) \otimes \rho_{f,\lambda_0})(\mathrm{Frob}_l)) &= \mathrm{tr}(((\chi_1 \oplus \chi_2) \otimes \rho_{f,\lambda})(\mathrm{Frob}_l)) \\
&\equiv \mathrm{tr}((\rho_{0,\lambda} \otimes \rho_{1,\lambda})(\mathrm{Frob}_l)) \mod \mathfrak{m}_\lambda \\
&\equiv \mathrm{tr}((\rho_{0,\lambda_0} \otimes \rho_{1,\lambda_0})(\mathrm{Frob}_l)) \mod \lambda,
\end{aligned}$$

for infinitely many λ. Note that the last congruence is mod λ in stead of mod $\mathfrak{m}_\lambda$ because both $(\chi_1 \oplus \chi_2) \otimes \rho_{f,\lambda_0}$ and $\rho_{\lambda_0} \simeq \rho_{0,\lambda_0} \otimes \rho_{1,\lambda_0}$ have coefficients in E. Hence the semi-simplification of $\rho_{0,\lambda_0}\otimes\rho_{1,\lambda_0}$ is $(\chi_1\oplus\chi_2)\otimes\rho_{1,\lambda_0}$. But this contradicts the assumption that $\rho_\lambda\simeq\rho_{0,\lambda}\otimes\rho_{1,\lambda}$ is absolutely irreducible.

Finally, we may assume that the residual representation of $\rho_{0,\lambda}$ is absolutely irreducible for each $\lambda\in\Sigma'$. We apply Serre's conjecture (the strong form) for each $\bar\rho_{0,\lambda}$ with $\lambda\in\Sigma'$. Then there exists $g_\lambda\in S_p(N,\mathbb{C})$ such that $\bar\rho_{g_\lambda,\lambda'}\sim\bar\rho_{0,\lambda}$ with $p=\mathrm{rch}(\lambda)=\mathrm{rch}(\lambda')$. As indicated by Kisin in the proof of Theorem (1.3.1) and Remarks (1.1.3) in [Kis07] (in particular, we use [CV92] for $\lambda\nmid 2$), after deleting finitely many primes from Σ', there indeed exists a cusp form $f_\lambda\in S_1(N,\mathbb{C})$ such that $\bar\rho_{f_\lambda,\lambda''}\sim\bar\rho_{g_\lambda,\lambda'}\sim\bar\rho_{0,\lambda}$. Now we can play the same game as before: As $S_1(N,\mathbb{C})$ has only finitely many normalized eigenforms, by enlarging E, we may assume that there exists an eigenform f' such that $\bar\rho_{0,\lambda}\sim\bar\rho_{f',\lambda}$ for each $\lambda\in\Sigma'$. For each fixed $\lambda_0\in\Sigma'$ and a rational prime $l\neq\mathrm{rch}(\lambda)$ such that $l\notin S$, by compatibility of ρ_λ, we show

$$\begin{aligned}\mathrm{tr}((\rho_{0,\lambda_0}\otimes\rho_{1,\lambda_0})(\mathrm{Frob}_l)) &= \mathrm{tr}((\rho_{0,\lambda}\otimes\rho_{1,\lambda})(\mathrm{Frob}_l))\\ &\equiv \mathrm{tr}((\rho_{f',\lambda}\otimes\rho_{f,\lambda})(\mathrm{Frob}_l)) \mod \mathfrak{m}_\lambda\\ &= \mathrm{tr}((\rho_{f',\lambda_0}\otimes\rho_{f,\lambda_0})(\mathrm{Frob}_l)) \mod \lambda\end{aligned}$$

for $\lambda\in\Sigma'$. Note that the last congruence is mod λ instead of mod $\mathfrak{m}_\lambda$ because both $\rho_{\lambda_0}\simeq\rho_{0,\lambda_0}\otimes\rho_{1,\lambda_0}$ and $\rho_{f',\lambda_0}\otimes\rho_{f,\lambda_0}$ have coefficients in E. So $\rho_{0,\lambda_0}\otimes\rho_{1,\lambda_0}\sim\rho_{f',\lambda_0}\otimes\rho_{f,\lambda_0}$ and then ρ_{λ_0} is modular by the main theorem of [Ram00]. Therefore ρ_λ is modular for all λ.

Now it suffices to prove the statement of Step 1. For each (rational) prime $l\in S$ and a prime $\lambda\in\mathrm{Spec}(\mathcal{O}_E)$ with $p=\mathrm{rch}(\lambda)\neq l$, we obtain a local representation $\rho_\lambda: G_{\mathbb{Q}_l}\to\mathrm{GL}_4(E_\lambda)$. We denote I_l, I_l^w and k_λ the inertia subgroup, wild inertia subgroup of $G_{\mathbb{Q}_l}$ and the residue field of E_λ respectively. Set $\widetilde{H}_l:=\rho_\lambda(I_l^w)$ and assume that $p\geq 3$. Consider the reduction map $\bar\rho_\lambda:\rho_\lambda(G_{\mathbb{Q}_l})\hookrightarrow\mathrm{GL}_4(\mathcal{O}_{E_\lambda})\twoheadrightarrow\mathrm{GL}_4(k_\lambda)$. Since $l\neq p$, we easily see that $\bar\rho_\lambda$ restricted to $\widetilde{H}_l$ is injective. Hence $|\widetilde{H}_l|$ divides $|\mathrm{GL}_4(k_\lambda)|=q^6\prod_{i=1}^4(q^i-1)$, where $q:=p^f$ with $f:=[k_\lambda:\mathbb{Z}/p\mathbb{Z}]$. Letting $g=[E:\mathbb{Q}]$, we conclude that $|\widetilde{H}_l|$ divides $\prod_{i=1}^4(p^{ig}-1)$. Pick a positive integer a_l such that there exists a class $y_l\in(\mathbb{Z}/l^{a_l}\mathbb{Z})^\times$ satisfying $\prod_{i=1}^4(y_l^{ig}-1)\not\equiv 0 \mod l^{a_l}$. Now set

$$\Sigma:=\{\lambda\in\mathrm{Spec}(\mathcal{O}_E)|\mathrm{rch}(\lambda)\notin\{2\}\cup S \text{ and } \mathrm{rch}(\lambda)\equiv y_l \mod l^{a_l},\forall l\in S\}.$$

Hence Σ is a set of infinitely many primes and for each prime $\lambda\in\Sigma$, we have $\log_l(|\widetilde{H}_l|)\leq a_l-1$ for each $l\in S$.

From the above proof, we see that if there exists a number field E' such that $\rho_{i,\lambda}(G_{\mathbb{Q}})\subset\mathrm{GL}_2(E'_{\lambda'})$ for each $\lambda\in\mathrm{Spec}(\mathcal{O}_E)$ then one can easily bound the size of image of I_l^w, and then bound the conductor as in §4.9 in [Ser87]. Unfortunately, we do not know the existence of E' in priori as $\rho_{i,\lambda}$ is constructed abstractly in Theorem 2.2.1. In the following, we show that there exists a character χ_λ such that after replacing $\rho_{i,\lambda}$ by $\rho_{0,\lambda}\otimes\chi_\lambda$ and $\rho_{1,\lambda}\otimes\chi_\lambda^{-1}$ respectively, we can directly bound $\rho_{i,\lambda}(I_l^w)$ in terms of $\widetilde{H}_l$.

More precisely, Set $\widetilde{K}_l:=\mathrm{Ker}(\rho_\lambda)\cap I_l^w$, $K_l:=\mathrm{Ker}(\rho_{0,\lambda})\cap\widetilde{K}_l$ and $K'_l:=\mathrm{Ker}(\rho_{1,\lambda})\cap\widetilde{K}_l$. All representations in the following proposition are representations restricted to $G_{\mathbb{Q}_l}$.

Proposition 4.3.1. *There exists a character $\chi_{\lambda,l} : G_{\mathbb{Q}_l} \to \overline{\mathbb{Q}}_p^\times$ of finite image such that after replacing $\rho_{0,\lambda}$ and $\rho_{1,\lambda}$ by $\rho_{0,\lambda} \otimes \chi_{\lambda,l}$ and $\rho_{1,\lambda} \otimes \chi_{\lambda,l}^{-1}$ respectively, $\widetilde{K}_l = K_l = K'_l$.*

Proof. Let $\tilde{L}$ and L be the fixed fields of $\widetilde{K}_l$ and K_l respectively. We have seen that $[L : \tilde{L}]$ must be finite as before: Select a *finite* extension E' over E_λ such that $\rho_{0,\lambda}(G_{\mathbb{Q}_l}) \subset \mathrm{GL}_2(E')$. As $p = \mathrm{rch}(\lambda) \neq l$ and $p \geq 3$, we easily check $\rho_{0,\lambda}(I_l^w)$ and $\rho_\lambda(I_l^w)$ inject into $\mathrm{GL}_2(k')$ and $\mathrm{GL}_4(k')$ respectively where k' is the residue field of $\mathcal{O}_{E'}$. So $\Gamma := \mathrm{Gal}(L/\tilde{L})$ is finite. For any $g \in \Gamma$, $\rho_\lambda(g) = \rho_{0,\lambda}(g) \otimes \rho_{1,\lambda}(g)$ is trivial, so we conclude that $\rho_{0,\lambda}(g) = \eta(g) I_2$ and $\rho_{1,\lambda}(g) = \eta^{-1}(g) I_2$ with η a character $\Gamma \to \overline{\mathbb{Q}}_p^\times$. Note that Γ is a subgroup of I_l^w/K_l. We easily extend η to a character of I_l^w/K_l, and obtain a character $\chi_{\lambda,l} : G_{\mathbb{Q}_l} \to \overline{\mathbb{Q}}_p^\times$ so that $\chi_{\lambda,l}(K_l) = \{1\}$ and $\chi_{\lambda,l}(g) = \eta^{-1}(g)$ for any $g \in I_l^w$. So by replacing $\rho_{0,\lambda}$ and $\rho_{1,\lambda}$ by $\rho_{0,\lambda} \otimes \chi_{\lambda,l}$ and $\rho_{1,\lambda} \otimes \chi_{\lambda,l}^{-1}$, L might have changed but $\tilde{L}$ does not change, we get $[L : \tilde{L}] = 1$ and prove the proposition. □

Now by class field theory, there exists a character $\chi_\lambda : G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p^\times$ such that $\chi_\lambda|_{I_l} = \chi_{\lambda,l}|_{I_l}$ for each $l \in S$ and χ_λ is unramified at $l \notin S$. Now replace $\rho_{0,\lambda}$ and $\rho_{1,\lambda}$ by $\rho_{0,\lambda} \otimes \chi_\lambda$ and $\rho_{1,\lambda} \otimes \chi_\lambda^{-1}$ respectively. For each $l \in S$ and $\lambda \in \Sigma$, both $|\rho_{0,\lambda}(I_l^w)|$ and $|\rho_{1,\lambda}(I_l^w)|$ divide $|\widetilde{H}_l|$, and hence divide $l^{a_l - 1}$. Hence the statement in Step 1 follows Proposition 9 in [Ser87].

4.4. **Applications to Scholl representations.** Suppose we are given a compatible system of p-adic Galois representations $\mathcal{R} = \{\mathbb{Q}, S, \{Q_l[X]\}, \{\rho_p, r(\rho_p)\}, \mathbf{v} = \{0, 0, m, m\}\}$, where $\rho_p(G_{\mathbb{Q}}) \subset \mathrm{GO}_4(\mathbb{Q}_p)$ for each p. For example, the compatible system comes from a Scholl motive in the introduction. Let $M/\mathbb{Q}$ be a quadratic extension and V_p denote the underlying space of ρ_p. We say $\mathcal{R}$ admits a *multiplication by M* if for each p there is an injection $\iota_p : M \hookrightarrow \mathrm{End}_{\mathbb{Q}_p} V_p$ of $\mathbb{Q}$-algebras and $g(\iota_p(m)v) = \iota_p(m)g(v)$ for $g \in G_M$, $m \in M$ and $v \in V_p$. For examples of such compatible system, see §5 in [ALLL13]. Note many systems in [ALLL13] have *quaternion* multiplications.

We claim that $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} \rho_p$ restricting to G_M is reducible. In fact, if $M = \mathbb{Q}(\sqrt{d})$ then $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} V_p \simeq V_p^+ \oplus V_p^-$ where V_p^+ and V_p^- are eigenspaces of $\iota_p(\sqrt{d})$. It is easy to check that V_p^+ and V_p^- are G_M-stable. Proposition 3.4.2 implies that either ρ_p is induced by a character or $F = \mathbb{Q}$, equivalently $\rho_p(G_{\mathbb{Q}}) \subset \mathrm{SGO}_4(\overline{\mathbb{Q}}_p)$. Then Corollary 1.0.4 follows Corollary 1.0.3.

References

[AC89] James Arthur and Laurent Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton University Press, Princeton, NJ, 1989.

[ALLL13] A. O. L. Atkin, Wen-Ching Winnie Li, Tong Liu, and Ling Long, *Galois representations with quaternion multiplication associated to noncongruence modular forms*, Trans. Amer. Math. Soc. **365** (2013), no. 12, 6217–6242. MR 3105749

[Ber04] Laurent Berger, *An introduction to the theory of p-adic representations*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 255–292.

[BLGGT11] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, *Potential automorphy and change of weight*, Preprint, appear at Annals of Mathematics (2011).

[Cal] Frank Calegari, *Even Galois representations*, Note, avaliable at `http://www.math.northwestern.edu/~fcale/papers/FontaineTalk-Adjusted.pdf`.

[Cal12] ———, *Even Galois Representations and the Fontaine-Mazur conjecture II*, Journal of the American Mathematical Society **25** (2012), 533–554.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.

[Cli37] A. H. Clifford, *Representations induced in an invariant subgroup*, Ann. of Math. (2) **38** (1937), no. 3, 533–550. MR 1503352

[Con] Brian Conrad, *Lifting global representations with local properties*, Preprint, available at `http://math.stanford.edu/~conrad/`.

[CV92] Robert F. Coleman and José Felipe Voloch, *Companion forms and Kodaira-Spencer theory*, Invent. Math. **110** (1992), no. 2, 263–281. MR 1185584 (93i:11063)

[DFG04] Fred Diamond, Matthias Flach, and Li Guo, *The Tamagawa number conjecture of adjoint motives of modular forms*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 5, 663–727. MR 2103471 (2006e:11089)

[Die08] Luis Dieulefait, *How to facet a gemstone: from potential modularity to the proof of Serre's modularity conjecture*, Proceedings of the "Segundas Jornadas de Teoría de Números", Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 2008, pp. 135–152.

[DM13] Giovanni Di Matteo, *On admissible tensor products in p-adic Hodge theory*, Compos. Math. **149** (2013), no. 3, 417–429.

[Eme11] Matthew Emerton, *Local-global compatibility in the p-adic Langlands programme for $GL_{2/\mathbb{Q}}$*, Preprint, avaliable at `http://www.math.uchicago.edu/~emerton/preprints.html` (2011).

[Kis07] Mark Kisin, *Modularity of 2-dimensional Galois representations*, Current developments in mathematics, 2005, Int. Press, Somerville, MA, 2007, pp. 191–230. MR 2459302 (2010a:11098)

[Kis09] ———, *The Fontaine-Mazur conjecture for* GL_2, J. Amer. Math. Soc. **22** (2009), no. 3, 641–690. MR 2505297 (2010j:11084)

[Pat13] Stefan Patrikis, *Variations on a theorem of tate*, Preprint, avaliable at `arXiv:1207.6724` (2013).

[Pat14] ———, *On the sign of regular algebraic polarizable automorphic representations*, Preprint, appear at Mathematische Annalen (2014).

[Ram] Dinakar Ramakrishnan, *An Exercise Concerning the Selfdual Cusp forms on* $\mathrm{GL}(3)$, Preprint, avaliable at `http://www.math.caltech.edu/people/dinakar.html`.

[Ram00] ———, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* $\mathrm{SL}(2)$, Ann. of Math. (2) **152** (2000), no. 1, 45–111.

[Ram02] ———, *Modularity of solvable Artin representations of* $\mathrm{GO}(4)$*-type*, Int. Math. Res. Not. (2002), no. 1, 1–54.

[Sav05] David Savitt, *On a conjecture of Conrad, Diamond, and Taylor*, Duke Math. J. **128** (2005), no. 1, 141–197.

[Sch85] A. J. Scholl, *Modular forms and de Rham cohomology; Atkin-Swinnerton-Dyer congruences*, Invent. Math. **79** (1985), no. 1, 49–77.

[Sch96] ———, *Vanishing cycles and non-classical parabolic cohomology*, Invent. Math. **124** (1996), no. 1-3, 503–524.

[Sen81] Shankar Sen, *Continuous cohomology and p-adic Galois representations*, Invent. Math. **62** (1980/81), no. 1, 89–116.

[Ser77] J.-P. Serre, *Modular forms of weight one and Galois representations*, Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), Academic Press, London, 1977, pp. 193–268.

[Ser87] Jean-Pierre Serre, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230.

[Ser89] ______, *Abelian l-adic representations and elliptic curves*, second ed., Advanced Book Classics, Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1989, With the collaboration of Willem Kuyk and John Labute.

[Ski09] Christopher Skinner, *A note on the p-adic galois representations attached to Hilbert modular forms*, Documenta Mathematica (2009), no. 14, 241–258.

Appendix A. Tensor being crystalline implies each factor being crystalline up to twist

LIANG XIAO

Let p be a prime number and K a finite extension of $\mathbb{Q}_p$. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of K and let $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ denote the absolute Galois group. For E a finite extension of $\mathbb{Q}_p$, we use $\mathbf{Rep}_E(G_K)$ to denote the category of continuous representations of G_K on a finite dimensional E-vector space.

Let $\mathbb{C}_p$ denote the p-adic completion of $\overline{\mathbb{Q}}_p$. We do not recall the definition of Fontaine rings $\mathbb{B}_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{cris}}$, etc, for which one may consult [1]. The aim of this appendix is to give a simpler proof of Theorem A.0.1 below, which is proved by Di Matteo in [4]. We hope that the proof presented here is more accessible to readers who are less familiar with p-adic Hodge theory, and hence makes Theorem A.0.5 at the end of the appendix less mysterious.

Theorem A.0.1. *Let $V, W \in \mathbf{Rep}_E(G_K)$ be two p-adic representations of G_K which are Hodge-Tate. If $W \otimes_E V$ is de Rham, then so are V and W themselves.*

Proof. We warn the readers that one has to be very careful when dealing with the coefficient fields. We start by some simple reductions dealing with coefficients. Since being de Rham or Hodge-Tate is preserved when replacing K and E by finite extensions, we may assume that $K = E$ is a Galois extension of $\mathbb{Q}_p$.

For $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$ and $U \in \mathbf{Rep}_K(G_K)$, set $\mathbf{D}_{\mathrm{dR},\sigma}(U) := (U \otimes_{K,\sigma} \mathbb{B}_{\mathrm{dR}})^{G_K}$, where $\otimes_{K,\sigma}$ means that the tensor is taken along the homomorphism $K \xrightarrow{\sigma} K \to \mathbb{B}_{\mathrm{dR}}$. We say U is *σ-de Rham* if $\dim_K \mathbf{D}_{\mathrm{dR},\sigma}(U) = \dim_K U$. Then U is de Rham if and only if it is σ-de Rham for all $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$. Theorem A.0.1 then follows Proposition A.0.4, which requires a few lemmas first. $\square$

Lemma A.0.2. *Assume that K is Galois over $\mathbb{Q}_p$. Let $V \in \mathbf{Rep}_K(G_K)$ be a Hodge-Tate representation. Then $\mathbf{D}_{\mathrm{dR},\sigma}(V) \neq 0$ for any $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$.*

Proof. We fix $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$. Since V is Hodge-Tate, we have $V \otimes_{K,\sigma} \mathbb{C}_p = \mathbb{C}_p(n_1) \oplus \cdots \oplus \mathbb{C}_p(n_d)$, for integers $n_1 \leq \cdots \leq n_d$. We first note that the basic properties of continuous cohomology of $\mathbb{C}_p(n)$ tell us

$$\mathrm{H}^i(G_K, V \otimes_{K,\sigma} t^n\mathbb{B}_{\mathrm{dR}}^+) = 0 \text{ for } i = 0, 1 \text{ and } n \gg 0.$$

Consider the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^0(G_K, V \otimes_{K,\sigma} t^{n+1}\mathbb{B}_{\mathrm{dR}}^+) \to \mathrm{H}^0(G_K, V \otimes_{K,\sigma} t^n\mathbb{B}_{\mathrm{dR}}^+) \to \mathrm{H}^0(G_K, V \otimes_{K,\sigma} \mathbb{C}_p(n)) \\ &\to \mathrm{H}^1(G_K, V \otimes_{K,\sigma} t^{n+1}\mathbb{B}_{\mathrm{dR}}^+) \to \mathrm{H}^1(G_K, V \otimes_{K,\sigma} t^n\mathbb{B}_{\mathrm{dR}}^+) \to \mathrm{H}^1(G_K, V \otimes_{K,\sigma} \mathbb{C}_p(n)). \end{aligned}$$

By an easy induction, we know that $\mathrm{H}^i(G_K, V \otimes_{K,\sigma} t^n\mathbb{B}_{\mathrm{dR}}^+) = 0$ for $i = 0, 1$ and $n > -n_1$. When $n = -n_1$, the first several terms of the long exact sequence above becomes

$$0 \to 0 \to \mathrm{H}^0(G_K, V \otimes_{K,\sigma} t^{-n_1}\mathbb{B}_{\mathrm{dR}}) \to \mathrm{H}^0(G_K, \mathbb{C}_p(n+n_1) \oplus \cdots \oplus \mathbb{C}_p(n+n_d)) \to 0.$$

Here, the relevant H^1-term vanishes because $n+1 > -n_1$. We see this forces $\mathrm{H}^0(G_K, V \otimes_{K,\sigma} t^{-n_1}\mathbb{B}_{\mathrm{dR}}) \neq 0$ and therefore $\mathbf{D}_{\mathrm{dR},\sigma}(V) \neq 0$. □

Lemma A.0.3. *If D is a $\mathbb{B}_{\mathrm{dR}}$-vector space of finite dimension d, with a semi-linear action of G_K. Then the natural map $D^{G_K} \otimes_K \mathbb{B}_{\mathrm{dR}} \to D$ is always injective. In particular, $\dim_K(D^{G_K}) \leq d$.*

Proof. This is a standard B-admissibility argument; for completeness, we reproduce it here. We need only to show that a K-linearly independent subset of D^{G_K} is also $\mathbb{B}_{\mathrm{dR}}$-linearly independent. Suppose not, we pick a counterexample of minimal number of (K-linearly independent) elements in D^{G_K}; in other words, $e_1, \dots, e_r \in D^{G_K}$ are K-linearly independent but $\alpha_1 e_1 + \cdots + \alpha_r e_r = 0$ for $\alpha_i \in \mathbb{B}_{\mathrm{dR}} \backslash \{0\}$. Since $\mathbb{B}_{\mathrm{dR}}$ is a field, we may moreover assume that $\alpha_1 = 1$. Applying $g \in G_K$ to this equality, we have

$$e_1 + g(\alpha_2)e_2 + \cdots + g(\alpha_r)e_r = 0 \quad \Rightarrow (g\alpha_2 - \alpha_2)e_2 + \cdots + (g\alpha_r - \alpha_r)e_r = 0.$$

By the minimality of the linear relation, we conclude that $g\alpha_i = \alpha_i$ for $i = 2, \dots, d$. Hence each $\alpha_i \in \mathbb{B}_{\mathrm{dR}}^{G_K} = K$. But $e_1, \dots, e_r$ were assumed to be K-linearly independent. We arrive at a contradiction. This proves the lemma. □

Proposition A.0.4. *Assume that K is Galois over $\mathbb{Q}_p$. Fix $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$. Let $V, W \in \mathbf{Rep}_K(G_K)$ be Hodge-Tate representations. If $W \otimes_K V$ is σ-de Rham, then so are V and W.*

Proof. Denote $n = \dim_K V$ and $m = \dim_K W$. By Lemma A.0.2, we know that $\mathbf{D}_{\mathrm{dR},\sigma}(V) = (V \otimes_{K,\sigma} \mathbb{B}_{\mathrm{dR}})^{G_K} \neq 0$. Let r denote the dimension of this K-vector space. Consider the quotient Q of the injective map (by Lemma A.0.3)

$$\mathbf{D}_{\mathrm{dR},\sigma}(V) \otimes_K \mathbb{B}_{\mathrm{dR}} \hookrightarrow V \otimes_{K,\sigma} \mathbb{B}_{\mathrm{dR}};$$

it is a vector space over $\mathbb{B}_{\mathrm{dR}}$ of dimension $n-r$ with continuous action of G_K. Now, taking the G_K-invariants of the following exact sequence

$$0 \to W \otimes_{K,\sigma} \mathbf{D}_{\mathrm{dR},\sigma}(V) \otimes_K \mathbb{B}_{\mathrm{dR}} \to (W \otimes_K V) \otimes_{K,\sigma} \mathbb{B}_{\mathrm{dR}} \to W \otimes_{K,\sigma} Q \to 0,$$

we obtain

$$0 \to \mathbf{D}_{\mathrm{dR},\sigma}(W) \otimes_K \mathbf{D}_{\mathrm{dR},\sigma}(V) \to \mathbf{D}_{\mathrm{dR},\sigma}(W \otimes_K V) \to \big(W \otimes_{K,\sigma} Q\big)^{G_K}.$$

By Lemma A.0.3, we know that the dimensions of the first and third terms are at most rm and $(n-r)m$ respectively; whereas the dimension of the middle term is nm because $W \otimes_K V$ is σ-de Rham. Therefore, $\dim_K \mathbf{D}_{\mathrm{dR},\sigma}(W) \otimes_K \mathbf{D}_{\mathrm{dR},\sigma}(V) = rm$, yielding W being σ-de Rham. A symmetric argument proves that V is also σ-de Rham. □

As a corollary of Theorem A.0.1, together with the well-known "black box": de Rham implies potentially semistable, it is not hard to deduce Theorem A.0.5 below.

Theorem A.0.5. *Let $V, W \in \mathbf{Rep}_E(G_K)$ be two p-adic representations of G_K. If $W \otimes_E V$ is crystalline, then there exists a finite extension F of E and a continuous character $\eta : G_K \to F^\times$ such that $V \otimes_F F(\eta)$ and $W \otimes_F F(\eta^{-1})$ are both crystalline, where $F(\eta)$ and $F(\eta^{-1})$ are 1-dimensional p-adic representations over F associated to η and η^{-1} respectively.*

Proof. The reduction to Theorem A.0.1 is carried out in [4]. For the convenience of the readers, we sketch the idea.

(As shown in [4, Section 2],) one can twist V and W by a character (coming from Lubin-Tate module of $\mathcal{O}_K$ to change the generalized Hodge-Tate weights (see [3]) to be integers; this is essentially because those Hodge-Tate weights from V and from W pairwise adds up to integers. One also easily see that the actions of the Tate-Sen operator on V and W are semisimple because their tensor product is. Hence, V and W are Hodge-Tate up to twists.

Now, by Theorem A.0.1, one concludes that, up to the same twist, V and W are de Rham. By the main theorem of [2], they are (up to twist) potentially semi-stable. Consider the associated Deligne-Weil representation. This question essentially reduces to representation question over $\mathbb{C}$-vector spaces, and is discussed in [4, Theorem 1.4]. □

References

[1] Laurent Berger, An introduction to the theory of p-adic representations, *Geometric aspects of Dwork theory*, Vol. I, 255–292, de Gruyter, Berlin, 2004.

[2] Laurent Berger, Représentations p-adiques et équations différentielles, *Inventiones mathematicae* **148** (2002), 219–284.

[3] Laurent Berger and Pierre Colmez, Familles de représentations de de Rham et monodromie p-adique, in *p-adic representations of p-adic groups I: Galois representations and (φ, Γ)-modules*, Astérisque **319** (2008), 303–337.

[4] Giovanni Di Matteo, On admissible tensor products in p-adic Hodge theory, *Compositio Mathematica* **149** (2013), 417–429.

Department of Mathematics, University of Connecticut, Storrs, CT 06269, U.S.A

E-mail address: liang.xiao@uci.edu

Department of Mathematics, Purdue University, Indiana, 47907, USA.

E-mail address: tongliu@math.purdue.edu

Wu Yee Sun College, The Chinese University of Hong Kong, Shatin, Hong Kong.

E-mail address: jkyu@ims.cuhk.edu.hk