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POTENTIALLY GL_2-TYPE GALOIS REPRESENTATIONS ASSOCIATED TO NONCONGRUENCE MODULAR FORMS

WEN-CHING WINNIE LI, TONG LIU, AND LING LONG

ABSTRACT. In this paper, we consider representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to modular forms for noncongruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. When the underlying modular curves have a model over $\mathbb{Q}$, these representations are constructed by Scholl in [Invent. Math. 99 (1985), pp. 49–77] and are referred to as Scholl representations, which form a large class of motivic Galois representations. In particular, by a result of Belyi, Scholl representations include the Galois actions on the Jacobian varieties of algebraic curves defined over $\mathbb{Q}$. As Scholl representations are motivic, they are expected to correspond to automorphic representations according to the Langlands philosophy. Using recent developments on automorphy lifting theorem, we obtain various automorphy and potential automorphy results for potentially GL_2-type Galois representations associated to noncongruence modular forms. Our results are applied to various kinds of examples. In particular, we obtain potential automorphy results for Galois representations attached to an infinite family of spaces of weight 3 noncongruence cusp forms of arbitrarily large dimensions.

1. Introduction

In a series of papers [LLY, ALL, Lon, HLV], the authors investigated 4-dimensional ℓ-adic representations of $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from noncongruence cusp forms constructed by Scholl [Sch1] which admit quaternion multiplication (QM) as given in Definition 4. In each case, it is shown that Galois representations are automorphic in the sense that they correspond to automorphic representations of GL_4 over $\mathbb{Q}$ as described by the Langlands program. (See Definition 3 for more details on automorphy and potential automorphy.) These automorphy results are obtained via the Faltings-Serre method which boils down to comparing the Galois representations associated with noncongruence modular forms and those attached to automorphic targets which were identified by an extensive search in the modular form database. In [AL^3], a general automorphy result for 4-dimensional Scholl representations admitting quaternion multiplication was obtained using the then newly established Serre conjecture and modularity lifting theorems. In the literature, there are many constructions of 4-dimensional Scholl representations with QM including families of 2-dimensional abelian varieties with QM (in the sense that their endomorphism algebra contains a quaternion algebra) which are parameterized by Shimura curves

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[Shi1, Del]. In [DFLST], the authors used hypergeometric functions over finite fields to study some natural families of Galois representations which are potentially of GL_2-type (see Defintion 1 below). In particular, they constructed a family of 8-dimensional ℓ-adic Galois representations $\{\rho_{\ell,t}\}$ of $G_{\mathbb{Q}}$ such that for each parameter $t \in \mathbb{Q} \smallsetminus \{0,1\}$ there is a Galois extension $F(t)/\mathbb{Q}$ depending on t and upon enlarging the scalar field the restrictions decompose as

$$\rho_{\ell,t}|_{G_{F(t)}} \cong \tilde{\sigma}_{\ell,t} \oplus \tilde{\sigma}_{\ell,t} \oplus \tilde{\sigma}_{\ell,t} \oplus \tilde{\sigma}_{\ell,t}.$$

Here and later G_F denotes the absolute Galois group of the field F, and $\tilde{\sigma}_{\ell,t}$ is an irreducible 2-dimensional ℓ-adic representation of $G_{F(t)}$. More details are given in §4.4.1. Representations behaving like $\rho_{\ell,t}$ are called potentially 2-isotypic (cf. Definition 1). For example, 4-dimensional Scholl representations with QM are potentially 2-isotypic. Such representations can be described quite explicitly using Clifford theory recalled in §2.6.

The aim of this paper is to establish the potential automorphy of Scholl representations potentially of GL_2-type or potentially 2-isotypic using recent important advances in automorphy lifting theorem. It should be pointed out that most of the known (potential) automorphy criteria are for regular representations, namely those with distinct Hodge-Tate weights. On the other hand, the Scholl representations attached to a d-dimensional space of cusp forms of weight $\kappa \geq 2$ for a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ have Hodge-Tate weights 0 and $1-\kappa$, each with multiplicity d (cf. [Sch1]). Hence they are highly irregular when $d > 1$. This paper is motivated by the hope that, for a Scholl representation potentially of GL_2-type, there is a good chance that the Hodge-Tate weights would be evenly distributed among the 2-dimensional subrepresentations so that the known criteria could be applied to conclude its potential automorphy.

Our main results are as follows.

Theorem A (Theorem 3.1). *Let F be a totally real field and let $\{\eta_\ell : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)\}$ be a system of 2-dimensional ℓ-adic Galois representations for all primes ℓ. Suppose there exist a finite extension K of F and a compatible system of Scholl representations $\{\rho_\ell\}$ such that for each ℓ, $\eta_\ell|_{G_K}$ is a subrepresentation of $\rho_\ell|_{G_K}$. Then there exists a set $\mathcal{L}$ of rational primes of Dirichlet density 1 so that for each $\ell \in \mathcal{L}$, η_ℓ is* totally odd*, i.e., $\det \eta_\ell(c) = -1$ for any complex conjugation $c \in G_F$, and potentially automorphic. Moreover, η_ℓ is automorphic if $F = \mathbb{Q}$ or η_ℓ is potentially reducible.*

Here a representation of a group G is called *potentially reducible* if it is reducible when restricted to a finite index subgroup of G.

As a consequence of Theorem A, one obtains sufficient conditions for (potential) automorphy of Scholl representations, Proposition 3.7, and Corollary 3.9, which are applied to examples studied in this paper. Theorem A is also used to prove the following.

Theorem B (Theorem 4.4). *Let $\{\rho_\ell\}$ be a compatible system of $2d$-dimensional semi-simple subrepresentations of Scholl representations of $G_{\mathbb{Q}}$. Assume that there is a finite Galois extension $F/\mathbb{Q}$ such that all ρ_ℓ are 2-isotypic when restricted to G_F. Suppose that F contains some solvable extension $K/\mathbb{Q}$ such that for each ℓ the representation $\rho_\ell \simeq \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \sigma_\ell$ for a 2-dimensional representation σ_ℓ. Then all ρ_ℓ are automorphic.*

So far the automorphy of degree-$2d$ Scholl representations attached to spaces of weight κ noncongruence cusp forms is known systematically for $d = 1$ and $\kappa \geq 2$ as a consequence of the Serre's conjecture established by Khare and Wintenberger [KW] and Kisin [Kis], and certain cases (e.g., GO_4-type) of $d = 2$ and odd $\kappa \geq 3$ obtained by Liu and Yu in [LY]. Other known automorphy results are also for low degree Scholl representations including [LLY, ALL, Lon, FHLRV, HLV, AL^3]. Applying the results in this paper, we prove the potential automorphy for an infinite family of explicitly constructed Scholl representations with unbounded degrees, extending the automorphy results shown in [LLY, ALL, Lon] alluded to above.

For $n \geq 2$ denote by $\rho_{n,\ell}$ the $2(n-1)$-dimensional ℓ-adic Scholl representation attached to the space of weight 3 cusp forms for the index-n normal subgroup Γ_n of the congruence subgroup $\Gamma^1(5)$ constructed in [ALL]. As explained in §5.3, the representation $\rho_{n,\ell}$ decomposes as

$$\rho_{n,\ell} = \bigoplus_{d|n,\ d>1} \rho_{d,\ell}^{new},$$

where $\rho_{n,\ell}^{new}$, as ℓ varies, form a compatible system of representations of $G_{\mathbb{Q}}$ of dimension $2\phi(n)$, and $\phi(n)$ is the degree of the cyclotomic field $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ with ζ_n a primitive nth root of unity.

Theorem C (Theorem 5.1)**.** *For $n \geq 2$, there are 2-dimensional ℓ-adic representations σ_ℓ of $G_{\mathbb{Q}(\zeta_n)}$ whose semi-simplifications form a compatible system, such that $\rho_{n,\ell}^{new} \cong \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{\mathbb{Q}}} \sigma_\ell$ for all primes ℓ. For each ℓ the representation $\rho_{n,\ell}^{new}$ is potentially automorphic. Further, it is automorphic if either $n \leq 6$ or σ_ℓ is potentially reducible.*

The (potential) automorphy results combined with the Atkin-Swinnerton-Dyer congruences satisfied by the coefficients of noncongruence modular forms obtained in [Sch1] provide a link between the Fourier coefficients of noncongruence modular forms with those of automorphic forms. A much less expected consequence is illustrated in [LL], where the automorphy of a 2-dimensional Scholl representation in turn shed new light on the arithmetic properties of the associated 1-dimensional space S_κ of noncongruence cusp forms of integral weight $\kappa \geq 2$. More precisely, the automorphy is a key ingredient to prove that any cusp form in S_κ with $\mathbb{Q}$-Fourier coefficients is a cusp form for a congruence subgroup if and only if its Fourier coefficients have bounded denominators. This result settles in part a longstanding conjecture in the area of noncongruence modular forms.

This paper is organized as follows. Basic notation and definitions are given in Section 2, where potential automorphy results for degree-2 Galois representations and Clifford theory are reviewed. Section 3 is devoted to Scholl representations and their 2-dimensional subrepresentations. Our goal is to use advances on automorphy lifting theorem to prove Theorem A and its consequences. Owing to the irregularity of Scholl representations and their tensors in general, the approach in [BGGT] cannot be applied directly. We pursue a variation by first using the results in [BGGT] to choose a suitable set $\mathcal{L}$ of rational primes of Dirichlet density 1; then for each $\ell \in \mathcal{L}$ such that η_ℓ is not potentially reducible (the difficult case), we study properties of the reduction of η_ℓ and its symmetric square, which lead to the proof of Theorem A by applying the known potential automorphy results summarized in Theorem 2.2. In Section 4, we study absolutely irreducible Scholl representations which are potentially of GL_1- or GL_2-type and prove Theorem B. We end this section with a few potentially 2-isotypic examples of (potentially) automorphic Scholl representations attached to weight 2 and weight 4 cusp forms, one of which was originally computed by Oliver Atkin. Theorem C is proved in Section 5 by realizing the Scholl representations $\rho_{n,\ell}$ as acting on the second étale cohomology of an elliptic surface $\mathcal{E}_n$ with an explicit model. When restricted to the Galois group of the cyclotomic field $\mathbb{Q}(\zeta_n)$, $\rho_{n,\ell}^{new}$ decomposes into the sum of a 2-dimensional subrepresentation $\sigma_{n,\ell}$ and its conjugates by $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. Using the symmetries on $\mathcal{E}_n$ we determine the trace of $\sigma_{n,\ell}$ (Theorem 5.3) from which it follows that the semi-simplifications of $\sigma_{n,\ell}$ are compatible as ℓ varies. The information on $\mathrm{Tr}\,\sigma_{n,\ell}$ implies that the degree-4 representation $\tau_{n,\ell}$ of the Galois group of the totally real subfield of $\mathbb{Q}(\zeta_n)$ induced from $\sigma_{n,\ell}$ is 2-isotypic over $\mathbb{Q}(\zeta_{2n})$ and Proposition 3.7 is then applied to conclude the potential automorphy of $\tau_{n,\ell}$ and hence $\rho_{n,\ell}$. As a byproduct of the trace computation, we obtain an estimate of certain character sums of Weil-type, Corollary 5.5. In §5.7 we show that $\tau_{n,\ell}$ admits QM over $\mathbb{Q}(\zeta_{2n})$, generalizing the known results for $n = 3, 4, 6$ discussed in [AL^3]. The QM structure provides an alternative approach to the potential automorphy of $\tau_{n,\ell}$ by appealing to results in [AL^3]. Finally we remark that the same method can be used to obtain similar potential automorphy results for several infinite families of Scholl representations attached to weight 3 cusp forms for noncongruence subgroups of $\Gamma_1(6)$ constructed in the work of Fang et al. [FHLRV].

2. Preliminaries

2.1. **Basic notation.** Let F be a number field and let v be a finite place of F. Denote by F_v the completion of F at v, $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ the *absolute Galois group* of F, in which the absolute Galois group $G_{F_v} := \mathrm{Gal}(\overline{F}_v/F_v)$ of F_v can be embedded. The inertia subgroup I_v at v consists of elements of G_{F_v} which induce the trivial action on the residue field of F_v. We use Fr_v to denote the conjugacy class in G_F of the *arithmetic Frobenius* at v which induces the Frobenius automorphism of the residue field of F_v, and use $\mathrm{Frob}_v := \mathrm{Fr}_v^{-1}$ to denote the *geometric Frobenius* at v. In this paper, an *ℓ-adic Galois representation* of $G = G_F$ or G_{F_v} is a continuous homomorphism from G to the group of $\overline{\mathbb{Q}}_\ell$-linear automorphisms of a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. We use $\overline{\mathbb{Q}}_\ell$ (instead of a finite extension of $\mathbb{Q}_\ell$) as the scalar field for convenience. By a *character* we mean a 1-dimensional representation. A representation of G_F or G_{F_v} is said to be *unramified* at v if it is trivial on the inertia subgroup I_v.

For brevity, we write $\sigma \otimes \eta$ for the tensor $\sigma \otimes_{\overline{\mathbb{Q}}_\ell} \eta$ of two ℓ-adic representations η and σ of the same group. Suppose F is Galois over $\mathbb{Q}$. Then for any representation ρ of G_F and $g \in G_{\mathbb{Q}}$, we can define another representation ρ^g of G_F, called the *conjugate* of ρ by g, via $\rho^g(h) := \rho(g^{-1}hg)$ for all $h \in G_F$. It is easy to check that, up to equivalence, ρ^g depends only on $g \in \mathrm{Gal}(F/\mathbb{Q})$.

2.2. GL_2-type and potentially 2-isotypic.

Definition 1. Let r be a positive integer. A continuous semi-simple ℓ-adic Galois representation ρ_ℓ of $G_{\mathbb{Q}}$ is said to be *potentially of* GL_r*-type* if there is a finite Galois extension $F/\mathbb{Q}$ such that the restriction $\rho_\ell|_{G_F}$ decomposes into a direct sum of r-dimensional irreducible subrepresentations. If, in addition, the r-dimensional subrepresentations are all isomorphic, ρ_ℓ is called *potentially r-isotypic*. When $F=\mathbb{Q}$, one simply says ρ_ℓ is *of* GL_r*-type* or *r-isotypic* accordingly.

In this paper we will be mostly concerned with *potentially of* GL_2*-type* and *potentially* 2*-isotypic* representations. As an example, consider the curve $C_{a,b}: y^2 = x^6+ax^4+bx^2+1$ with $a,b\in\mathbb{Q}$. For generic choices of $a,b\in\mathbb{Q}$, it has genus 2. Let $\rho_{\ell,a,b}$ denote the 4-dimensional ℓ-adic Galois representation of $G_{\mathbb{Q}}$ arising from the Tate module of ℓ-power torsion points on the Jacobian of $C_{a,b}$, which is known to be semi-simple. On $C_{a,b}$ there is the involution $\tau_1:(x,y)\mapsto(-x,y)$ defined over $\mathbb{Q}$. Its induced action on the representation space of $\rho_{\ell,a,b}$ decomposes the space into eigenspaces $\sigma_{\ell,a,b,\pm}$ with eigenvalues ±1, both invariant under the Galois action. Therefore $\rho_{\ell,a,b}\cong\sigma_{\ell,a,b,+}\oplus\sigma_{\ell,a,b,-}$, and hence is of GL_2-type. When $a=b$, there is another map $\tau_2:(x,y)\mapsto(\frac{1}{x},\frac{y}{x^3})$ on $C_{a,a}$ defined over $\mathbb{Q}$. It is straightforward to check that both τ_1 and τ_2 have order 2 and $\tau_1\tau_2\tau_1^{-1}\tau_2^{-1}=(\tau_1\tau_2)^2$ sends (x,y) to $(x,-y)$ which induces the multiplication by -1 map on the Jacobian of $C_{a,a}$. In other words, the induced actions of τ_1 and τ_2 on the Jacobian of $C_{a,a}$ anticommute with each other, thus τ_2 intertwines the two representations $\sigma_{\ell,a,b,+}$ and $\sigma_{\ell,a,b,-}$ so that they are isomorphic 2-dimensional representations of $G_{\mathbb{Q}}$. This makes $\rho_{\ell,a,a}$ a prototype of 2-isotypic representations. Observe that $C_{a,a}$ is a two-fold cover of the elliptic curve $E_a: y^2=x^3+ax^2+ax+1$ (assuming the discriminant of the curve is nonzero) which gives rise to a compatible system of 2-dimensional ℓ-adic representations $\sigma_{\ell,a}$ of $G_{\mathbb{Q}}$. Thus $\sigma_{\ell,a}$ is isomorphic to a subrepresentation of $\rho_{\ell,a,a}$. As $\rho_{\ell,a,a}$ is 2-isotypic, one concludes that $\rho_{\ell,a,a}\cong\sigma_{\ell,a}\oplus\sigma_{\ell,a}$. Later in §4.4.1 we will see similar examples with F being nontrivial extensions of $\mathbb{Q}$.

2.3. **Local properties and τ-Hodge-Tate weights.** Let F be a number field and let ρ be an ℓ-adic representation of G_F. In this subsection we discuss the local property of ρ at a place v of F dividing ℓ via the ℓ-adic Hodge theory. The reader is referred to [Lau] for basic notions such as crystalline representations, de Rham representations, Hodge-Tate weights, etc., in ℓ-adic Hodge theory. Recall that an ℓ-adic Galois representation $\rho: G_F\to\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is called *geometric* if ρ is unramified almost everywhere and $\rho|_{G_{F_v}}$ is de Rham for each prime v of F dividing ℓ. For the remainder of this paper, we always assume an ℓ-adic Galois representation is geometric unless otherwise specified.

Each place v of F dividing ℓ corresponds to an embedding $\tau: F\to\overline{\mathbb{Q}}_\ell$, and τ extends to an embedding of F_v in $\overline{\mathbb{Q}}_\ell$. The *$\tau$-Hodge-Tate weights* of ρ is defined to be the multiset $\mathrm{HT}_\tau(\rho)$ consisting of the integers i with multiplicity equal to

$$\dim_{\overline{\mathbb{Q}}_\ell}(\rho|_{G_{F_v}}\otimes_{\tau,F_v}\mathbb{C}_\ell(-i))^{G_{F_v}}.$$

Here $\mathbb{C}_\ell$ denotes the completion of $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}_\ell(-i)$ is the usual notation for Tate twists.

For example, if $\rho=\epsilon_\ell$ is the ℓ-adic cyclotomic character, whose value at a geometric Frobenius element Frob_w at a finite place w of F not dividing ℓ is the inverse of the residual cardinality at w, then $\mathrm{HT}_\tau(\epsilon_\ell)=\{1\}$. When $F=\mathbb{Q}$, the trivial embedding τ will be omitted from the notation HT_τ. Collected below are some useful facts concerning Hodge-Tate weights.

Lemma 2.1. *Let ρ and γ be two ℓ-adic Galois representations of G_F. Then*

(1) $\mathrm{HT}_\tau(\rho\otimes\gamma)=\{a_\tau+b_\tau | a_\tau\in\mathrm{HT}_\tau(\rho), b_\tau\in\mathrm{HT}_\tau(\gamma)\}$.

(2) *Suppose that $F/\mathbb{Q}$ is Galois. Then* $\mathrm{HT}_\tau(\rho^g) = \mathrm{HT}_{\tau g^{-1}}(\rho)$ *for any* $g \in \mathrm{Gal}(F/\mathbb{Q})$.

(3) *Suppose that F is totally real and ρ is a 1-dimensional geometric ℓ-adic representation. Then* $\mathrm{HT}_\tau(\rho)$ *is independent of the embedding* $\tau : F \to \overline{\mathbb{Q}}_\ell$.

Proof. (1) It can be easily checked from the definition.

(2) Let $\tilde{g} \in G_{\mathbb{Q}}$ be a lift of g. We easily check that the map $\sum_i v_i \otimes a_i \mapsto \sum_i v_i \otimes \tilde{g}(a_i)$ induces a $\overline{\mathbb{Q}}_\ell$-linear bijection between $(\rho|_{G_{F_v}} \otimes_{\tau, F_v} \mathbb{C}_\ell(-i))^{G_{F_v}}$ and $(\rho|_{G_{F_{g(v)}}} \otimes_{\tau, F_{g(v)}} \mathbb{C}_\ell(-i))^{G_{F_{g(v)}}}$. Then the claim follows.

(3) Since F is totally real by assumption, so is its maximal CM (complex multiplication) subfield. We apply the discussion before Lemma A.2.1 of [BGGT] to conclude that $\mathrm{HT}_\tau(\rho)$ (which is a singleton) does not depend on any embedding $\tau : F \to \overline{\mathbb{Q}}_\ell$. □

2.4. Compatible system of Galois representations and automorphy.

Definition 2. Let F be a number field. For each finite place v of F, denote by $\mathrm{rch}(v)$ the residual characteristic of v. Following [BGGT], by *a rank n compatible system of ℓ-adic Galois representations $\mathcal{R}$ of G_F defined over E* we mean a quadruple

$$\{E, S, \{Q_\mathfrak{p}(X)\}, \{\rho_\lambda\}\}$$

where

1. E is a number field;
2. S is a finite set of places of F;
3. for each finite place $\mathfrak{p}$ of F outside S, $Q_\mathfrak{p}(X)$ is a monic degree n polynomial in $E[X]$;
4. for each finite place λ of E,

$$\rho_\lambda : G_F \longrightarrow \mathrm{GL}_n(\overline{E}_\lambda)$$

is a continuous semi-simple representation such that

- ρ_λ is unramified at finite places $\mathfrak{p}$ of F outside S with $\mathrm{rch}(\mathfrak{p}) \neq \mathrm{rch}(\lambda)$, and $\rho_\lambda(\mathrm{Frob}_\mathfrak{p})$ has characteristic polynomial $Q_\mathfrak{p}(X)$.

If we further assume

- $\rho_\lambda|_{G_{F_\mathfrak{p}}}$ is de Rham at finite places $\mathfrak{p}$ of F with $\mathrm{rch}(\mathfrak{p}) = \mathrm{rch}(\lambda)$, and further, it is crystalline if $\mathfrak{p} \notin S$,

and the quadruple satisfies

- for each prime ℓ and each embedding $\tau : F \to \overline{\mathbb{Q}}_\ell$, there exists a fixed set v_τ of integers such that $\mathrm{HT}_\tau(\rho_\lambda) = \mathrm{v}_\tau$ for all places λ of E dividing ℓ,

then we call the quintuple $\{E, S, \{Q_\mathfrak{p}[X]\}, \{\rho_\lambda\}, \{\mathrm{v}_\tau\}\}$ a *strongly compatible system.*

We warn readers that the above definition differs from that in [BGGT]: the strongly compatible system here is called the *weakly compatible system* in [BGGT]. Here we follow the terminology in the classical setting by Serre in [Ser1]. The properties of ρ_λ at primes above ℓ were added to the definition of the compatible system in [Ser1] only in the recent decade. We adapt the above version which is convenient for our purposes.

Definition 3. An ℓ-adic Galois representation $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is called *automorphic* if there exist an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and an automorphic representation $\pi \simeq \bigotimes'_v \pi_v$ of $\mathrm{GL}_n(\mathbb{A}_F)$ such that for almost all primes v of F the (Frobenius-semi-simplification of the) Weil-Deligne representation associated to $\rho|_{G_{F_v}}$ via ι is isomorphic to the Weil-Deligne representation associated to π_v as described by the local Langlands correspondence. (See, for instance, [BGGT] for details.) In particular, ι sends the eigenvalues of the characteristic polynomial of $\rho(\mathrm{Frob}_v)$ to the Satake parameters of π_v for almost all places v of F. We call ρ *potentially automorphic* if there exists a finite extension F' of F so that $\rho|_{G_{F'}}$ is automorphic.

Let F' be a soluble extension of F. According to the solvable base change in [AC] by Arthur and Clozel, if a representation ρ of G_F is automorphic, then so is its restriction $\rho|_{G_{F'}}$. Conversely, if a representation σ of $G_{F'}$ is automorphic, then so is the induced representation $\mathrm{Ind}_{G_{F'}}^{G_F}\sigma$ of G_F.

2.5. **Potential automorphy results for degree-2 Galois representations.** We summarize the known (potential) automorphy results for 2-dimensional Galois representations which will be used later in the paper. Let F be a totally real field and let $\eta : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be an ℓ-adic Galois representation. We call η (totally) *odd* if $\det(\eta)(c) = -1$ for any complex conjugation $c \in G_F$. For any finite-dimensional ℓ-adic representation σ of G_F, its ambient space always contains a G_F-stable $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$-lattice L. Here $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$ stands for the ring of integers of $\overline{\mathbb{Q}}_\ell$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$. Then σ induces an action $\bar{\sigma}$ of G_F on the quotient space $L/\mathfrak{m}L$ over the residue field $\overline{\mathbb{F}}_\ell$, called the *reduction* of σ. It is well known that the semi-simplification of $\bar{\sigma}$ does not depend on the choice of the lattice L.

Theorem 2.2. *Let F be a totally real field and let $\eta : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a continuous representation. Assume the following:*

(a) *η is irreducible and unramified almost everywhere;*
(b) *F is unramified at ℓ; for each prime v of F above ℓ, $\eta|_{G_{F_v}}$ is crystalline and for each embedding $\tau : F \to \overline{\mathbb{Q}}_\ell$, $\mathrm{HT}_\tau(\eta) = \{a_\tau, a_\tau + b_\tau\}$ with $0 < b_\tau < (\ell - 1)/2$;*
(c) *$\mathrm{Sym}^2\bar{\eta}|_{G_{F(\zeta_\ell)}}$ is irreducible;*
(d) *$\ell > 7$.*

Then the following statements hold:

(1) *There is a finite totally real Galois extension F'/F such that $\eta|_{G_{F'}}$ is automorphic.*
(2) *Given finitely many 2-dimensional ℓ-adic representations $\eta_i, i = 1, \cdots, m$, of G_F, if each η_i satisfies the above assumptions* (a)–(d)*, then there exists a finite totally real Galois extension F' of F, depending on the representations, such that $\eta_i|_{G_{F'}}$ is automorphic for each $1 \le i \le m$.*
(3) *If $F = \mathbb{Q}$, then η is automorphic* (*modular*).

Proof. The assumption (c) implies the irreducibility of $\bar{\eta}|_{G_{F(\zeta_\ell)}}$. Together with the assumption (d) we conclude that η is totally odd from [CG, Prop. 2.5]. The assumption (b) implies that $\eta|_{G_{F_v}}$ is potentially diagonalizable so that a required condition to apply Theorem C of [BGGT] is satisfied. Together with Lemma 1.4.3 (iii) in [BGGT], we see that the assertion (1) is the special case of Theorem C in [BGGT] for $n = 2$.

For (2), we first warn the reader that this is not a (formal) consequence of (1), because it is not known that the automorphy of $\eta|_{G_{F'}}$ would imply the automorphy of $\eta|_{G_{F''}}$ for *any* finite totally real extension F''/F'. To prove (2), we use [BGGT, Thm. 4.5.1] and the idea of the proof of Corollary 4.5.2 in [BGGT]. Pick a totally imaginary quadratic extension M/F which is linearly disjoint from $K(\zeta_\ell)$ over F, where K is a finite extension contained in all splitting fields of $\mathrm{Sym}^2\bar{\eta}_i$, namely the field fixed by the kernel of $\mathrm{Sym}^2\bar{\eta}_i$. As observed above, each η_i is totally odd. Then $(\eta_i|_{G_M}, \det\eta_i)$ satisfies the assumption of Theorem 4.5.1 in [BGGT] so that there exists a finite Galois CM extension M_1/M such that $(\eta_i|_{G_{M_1}}, \det\eta_i|_{G_{F'}})$ is automorphic for all i, where F' is the maximal totally real subfield of M_1. Then $\eta_i|_{G_{F'}}$ is also automorphic by Lemma 2.2.2 in [BGGT] for all i.

If $F = \mathbb{Q}$, then the main result in [DFG] together with the input of Serre's conjecture and oddness of η implies that η is modular. This proves (3). $\square$

2.6. **Clifford theory in the context of Galois representations.** We end this section by summarizing some useful results in [Cli] in the context of Galois representations. Let F and k be fields. Denote by

$$\pi : \mathrm{GL}_n(k) \twoheadrightarrow \mathrm{PGL}_n(k)$$

the natural projection. Two Galois representations $\tau, \tau' : G_F \to \mathrm{GL}_n(k)$ are said to be *projectively equivalent* if there exists an invertible matrix $A \in \mathrm{GL}_n(k)$ such that

$$\pi \circ \tau = \pi \circ (A\tau' A^{-1}).$$

This is equivalent to the existence of a character $\chi : G_F \to k^\times$ so that $\tau \simeq \chi \otimes \tau'$. If χ is a character of finite image, then τ and τ' are called *finitely projectively equivalent.*

The next useful theorem follows from [Cli] and Tate's result on the vanishing of Galois cohomology [Tat] or [Ser2, §6.5].

Let k be an algebraically closed field and let $\rho : G_F \to \mathrm{GL}_n(k)$ be an irreducible representation. Given a finite Galois extension L/F, decompose the restriction of ρ to G_L into a direct sum of irreducible representations of G_L:

$$\rho|_{G_L} \simeq \sigma_1 \oplus \cdots \oplus \sigma_m.$$

Write σ for σ_1 and set $H := \{g \in G_F | \sigma^g \simeq \sigma\}$. Denote by $M := \overline{\mathbb{Q}}^H$ the fixed field of H. Note that H contains G_L and M is a subfield of L containing F.

Theorem 2.3. *Under the above setting, the following statements hold:*

(1) *For each $i = 1, \ldots, m$ there exists an element $g(i) \in \mathrm{Gal}(L/F)$ such that $\sigma_i \simeq \sigma^{g(i)}$. Consequently the σ_i's have the same dimension and $[M : F]$ is equal to the number of nonisomorphic σ_i's.*

(2) *There exist representations $\eta : G_M \to \mathrm{GL}_r(k)$ and $\gamma : G_M \to \mathrm{GL}_s(k)$ such that*

  (2a) *$\eta|_{G_L}$ is finitely projectively equivalent to σ, and γ has finite image such that $\gamma|_{G_L}$ is finitely projectively equivalent to s copies of the trivial representation of G_L.*

  (2b) *$\rho \simeq \mathrm{Ind}_{G_M}^{G_F}(\gamma \otimes \eta)$.*

For the sake of self-containedness, we sketch a (slightly different) proof here. Let V denote the underlying k-space of ρ. Define W to be the subspace of V spanned by σ^g for all $g \in H$. Set $H' := \{g \in G_F | g(W) = W\}$. Obviously, W

is a representation of H'. Theorem 2 in [Cli] shows that $V = \mathrm{Ind}_{H'}^{G_F} W$. Now we prove that $H' = H = G_M$. Obviously, $H \subset H'$ by definition. For any $g \in H'$, $\sigma^g(W) \subset W$ by definition. But W is a direct sum of irreducible representations isomorphic to σ. So $\sigma^g \simeq \sigma$ and hence $g \in H$. Therefore $V = \mathrm{Ind}_{G_M}^{G_F} W$. Since $W|_{G_L}$ is r-isotypic by the construction of W, Theorem 3 in [Cli] shows that there exist projective representations $\tilde{\gamma} : G_M \to \mathrm{PGL}_s(k)$ of G_M/G_L, and $\tilde{\eta} : G_M \to \mathrm{PGL}_r(k)$ so that $W \simeq \tilde{\gamma} \otimes \tilde{\eta}$, where $\tilde{\gamma}|_{G_L}$ is projectively equivalent to s copies of the trivial representation of G_L and $\tilde{\eta}|_{G_L}$ is projectively equivalent to σ. By Tate's result on vanishing of Galois cohomology, $\tilde{\gamma}$ admits a lifting $\gamma : G_M \to \mathrm{GL}_s(k)$ with finite image. Hence $\tilde{\eta}$ also has a lifting $\eta : G_M \to \mathrm{GL}_r(k)$ such that $W \simeq \gamma \otimes \eta$. This proves the theorem.

3. Galois representations arising from noncongruence cusp forms and their 2-dimensional subrepresentations

3.1. **Scholl representations associated to noncongruence cusp forms.** Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a *noncongruence* subgroup, that is, Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ not containing any principal congruence subgroup $\Gamma(N)$. For any integer $\kappa \geq 2$, the space $S_\kappa(\Gamma)$ of weight κ cusp forms for Γ is finite-dimensional; denote by $d = d(\Gamma, \kappa)$ its dimension. Assume that the compactified modular curve $\Gamma\backslash\mathfrak{H}^*$ (by adding cusps) is defined over $\mathbb{Q}$ and the cusp at infinity is $\mathbb{Q}$-rational. For even $\kappa \geq 4$ and any prime ℓ, in [Sch1] Scholl constructed an ℓ-adic Galois representation $\rho_\ell : G_\mathbb{Q} \to \mathrm{GL}_{2d}(\mathbb{Q}_\ell)$ attached to $S_\kappa(\Gamma)$. It turns out that there exist a finite set S of primes, polynomials $Q_p(X) \in \mathbb{Z}[X]$ for $p \notin S$, and a finite set v so that $\{\mathbb{Q}, S, \{Q_p(X)\}, \{\rho_\ell\}, \mathrm{v}\}$ form a *strongly compatible system.*[1] Here $\mathrm{v} = \mathrm{HT}(\rho_\ell) = \{0, \dots, 0, 1-\kappa, \dots, 1-\kappa\}$ consists of $1-\kappa$ and 0, each with multiplicity d, and is independent of ℓ. Scholl also showed that all the roots of $Q_p(X)$ have the same complex absolute value $p^{(\kappa-1)/2}$ (cf. §5.3 in [Sch1]). These results of Scholl can be extended to odd weights under some extra hypotheses (e.g., $\pm(\Gamma \cap \Gamma(N)) = \pm(\Gamma) \cap \pm(\Gamma(N))$ for some $N \geq 3$, where $\pm : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$ is the projection). The reader is referred to the end of [Sch1] for more details. In this paper we assume that ρ_ℓ exists.

Let V_ℓ denote the underlying $\mathbb{Q}_\ell$-space of ρ_ℓ. There exists a perfect, Galois action compatible pairing
$$V_\ell \times V_\ell \to \mathbb{Q}_\ell(-\kappa+1)$$
which is alternating (resp., symmetric) when κ is even (resp., odd). In particular, we have $\rho_\ell^\vee \simeq \epsilon_\ell^{\kappa-1}\rho_\ell$ for the dual representation $\rho_\ell^\vee$ of ρ_ℓ, where ϵ_ℓ denotes the ℓ-adic cyclotomic character.

For the remainder of the paper, we reserve ρ_ℓ for the ℓ-adic Galois representation associated to a noncongruence subgroup and call it a *Scholl representation* if no confusion arises. As explained in the introduction, Scholl representations are expected to correspond to certain automorphic forms. But since they are irregular when $d > 1$, the currently known (potential) automorphy lifting theorem cannot be applied directly. In this section, we will show that Scholl representations potentially of GL_2-type are (potentially) automorphic. See Theorem 3.1 for the precise statement.

[1] It is unclear that ρ_ℓ is always semi-simple from Scholl's construction. So in the following, we always replace ρ_ℓ by its semi-simplification if needed.

A (general) representation σ of a Galois group G_F is said to be *potentially reducible* if there exists a finite index subgroup H of G_F such that $\sigma|_H$ is reducible; otherwise, it is called *strongly irreducible.* Recall that a 2-dimensional representation σ of the Galois group of a totally real field F is *totally odd* if $\det \sigma(c) = -1$ for any complex conjugation $c \in G_F$.

The goal of Section 3 is to prove the following.

Theorem 3.1. *Let F be a totally real field and let $\eta_\ell : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a system of 2-dimensional ℓ-adic Galois representations. Suppose there exist a finite extension K of F and a compatible system of Scholl representations $\{\rho_\ell\}$ such that for each ℓ, $\eta_\ell|_{G_K}$ is a subrepresentation of $\rho_\ell|_{G_K}$. Then there exists a set $\mathcal{L}$ of rational primes of Dirichlet density 1 so that for each $\ell \in \mathcal{L}$, η_ℓ is totally odd and potentially automorphic. Moreover, η_ℓ is automorphic if $F = \mathbb{Q}$ or η_ℓ is potentially reducible.*

The proof will be divided into two parts according to whether η_ℓ is potentially reducible or strongly irreducible.

3.2. **Potentially reducible η_ℓ in Theorem 3.1.** We begin by exploring general properties of η_ℓ in Theorem 3.1, including its determinant, irreducibility, and Hodge-Tate weights.

Proposition 3.2. *Let F be a totally real field and let $\eta_\ell : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a Galois representation. Suppose that there exists a finite extension K/F so that $\eta_\ell|_{G_K}$ is isomorphic to a subrepresentation of $\rho_\ell|_{G_K}$ for a Scholl representation ρ_ℓ of $G_\mathbb{Q}$ associated to a space of cusp forms of weight $\kappa \geq 2$. Then η_ℓ is (absolutely) irreducible, $\det \eta_\ell = \epsilon_\ell^{1-\kappa}\chi$ for a character χ of finite order, and $\mathrm{HT}_\tau(\eta_\ell) = \{0, -\kappa+1\}$ for all embeddings $\tau : F \to \overline{\mathbb{Q}}_\ell$.*

Proof. We first prove the statement on $\det \eta_\ell$ and Hodge-Tate weights. Let τ be an embedding of F into $\overline{\mathbb{Q}}_\ell$. Note that $\mathrm{HT}_\tau(\eta_\ell) = \mathrm{HT}_{\tau'}(\eta_\ell|_{G_K})$ for any embedding $\tau' : K \to \overline{\mathbb{Q}}_\ell$ extending τ. Since $\eta_\ell|_{G_K}$ is a subrepresentation of $\rho_\ell|_{G_K}$, we see that $\mathrm{HT}_\tau(\eta_\ell)$ only has three possibilities: $\{0, 0\}$, $\{-\kappa+1, -\kappa+1\}$, or $\{0, -\kappa+1\}$. Then the determinant of η_ℓ has $\mathrm{HT}_\tau(\det \eta_\ell) = \{r\}$ with $r = 0$ or $2 - 2\kappa$ or $1 - \kappa$, equal to the sum of the two weights of η_ℓ accordingly. Since F is totally real, by Lemma 2.1, $\mathrm{HT}_\tau(\det \eta_\ell)$ is the same for all τ. Hence $\det \eta_\ell = \epsilon_\ell^r \chi$ for some character χ with Hodge-Tate weight 0. Since χ has Hodge-Tate weights 0 for all primes $\mathfrak{p}|\ell$, we have that $\chi(I_\mathfrak{p})$, the image of the inertia group at $\mathfrak{p}$, is a finite group, by Proposition 3.56 of [FO]. For any finite prime $\mathfrak{p} \nmid \ell$, $\chi(I_\mathfrak{p})$ is also a finite group. This is because the wild ramification group must have a finite image, and the same holds for the tame ramification group, resulting from the relation between a Frobenius element and a generator of the tame ramification group and the fact that χ is a character. As χ is ramified at finitely many places, we conclude the existence of a positive integer n such that χ^n is a character unramified at *all* places. By class field theory χ^n has finite order, therefore so has χ.

As a subrepresentation of $\rho_\ell|_{G_K}$, at almost all finite places $\mathfrak{p}$ of K, the roots of the characteristic polynomial of $\eta_\ell(\mathrm{Frob}_\mathfrak{p})$ have the same complex absolute value $q^{\frac{\kappa-1}{2}}$, where q is the cardinality of the residue field of $\mathfrak{p}$. Hence the complex absolute value of $\det \eta_\ell(\mathrm{Frob}_\mathfrak{p})$ is $q^{\kappa-1}$. On the other hand, the complex absolute value of $\epsilon_\ell(\mathrm{Frob}_\mathfrak{p})$ is q^{-1} and that of $\chi(\mathrm{Frob}_\mathfrak{p})$ is 1 since χ has finite order so that $\det \eta_\ell =$

$\epsilon_\ell^r\chi$ at $\mathrm{Frob}_\mathfrak{p}$ has complex absolute value q^{-r}. This proves that $r = 1-\kappa$ and $\mathrm{HT}_\tau(\eta_\ell) = \{0, -\kappa+1\}$ for all τ.

Now suppose that η_ℓ is reducible. Then the semi-simplification of η_ℓ is $\zeta \oplus \zeta'$ for some 1-dimensional λ-adic representations ζ and ζ' of G_F. By Lemma 2.1 again and replacing ζ by ζ' if necessary, we may assume that $\mathrm{HT}_\tau(\zeta) = 0$ for all τ and $\mathrm{HT}_\tau(\zeta') = 1-\kappa$ for all τ. The above argument shows that $\zeta = \chi'$ and $\zeta' = \epsilon_\ell^{1-\kappa}\chi''$ for some finite order characters χ' and χ''. The characteristic polynomial of $\eta_\ell(\mathrm{Frob}_\mathfrak{p})$ at an unramified finite place $\mathfrak{p}$ of K is $(X - \chi'(\mathrm{Frob}_\mathfrak{p}))(X - \epsilon_\ell^{1-\kappa}(\mathrm{Frob}_\mathfrak{p})\chi''(\mathrm{Frob}_\mathfrak{p}))$ with two roots of different complex absolute values, contradicting Scholl's result. So η_ℓ must be irreducible. □

Next we show that if η_ℓ in Theorem 3.1 is potentially reducible, then it is totally odd and automorphic.

Proposition 3.3. *Suppose that η_ℓ in Proposition 3.2 is potentially reducible. Then η_ℓ is totally odd, and there is a quadratic CM extension field M of F and a character χ_1 of G_M such that $\eta_\ell = \mathrm{Ind}_{G_M}^{G_F}\chi_1$. Consequently η_ℓ is automorphic.*

Proof. It follows from the proposition above that η_ℓ is irreducible with $\mathrm{HT}_\tau(\eta_\ell) = \{0, -\kappa+1\}$ for all embeddings $\tau : F \to \overline{\mathbb{Q}}_\ell$. By assumption, there is a finite extension L of F such that $\eta_\ell|_{G_L}$ is reducible. Then by Theorem 2.3, $\eta_\ell|_{G_L} \cong \chi_1 \oplus \chi_2$ with distinct characters χ_i of G_L, and furthermore, there exists a quadratic extension M of F so that χ_1 can be extended to a character of G_M and $\eta_\ell = \mathrm{Ind}_{G_M}^{G_F}\chi_1$. It turns out that M has to be a CM field, for otherwise χ_1 would be a power of the ℓ-adic cyclotomic character twisted by some finite character (see Proposition 1.12 of [Far]) and this contradicts the fact that η_ℓ has two distinct Hodge-Tate weights $\{0, 1-\kappa\}$. This shows that η_ℓ is totally odd. As χ_1 is geometric, it is well known that χ_1 is automorphic (see for example [Far]), hence so is η_ℓ by quadratic automorphic induction [AC]. □

3.3. **A proof of Theorem 3.1.** In view of the previous subsection, the heart of the proof of Theorem 3.1 is to handle the strongly irreducible η_ℓ's. Owing to the irregularity of Scholl representations and their tensors in general, the approach in [BGGT] by Barnet-Lamb, Gee, Geraghty, and Taylor cannot be applied directly. We pursue a variation as follows. First, using the results in [BGGT] we choose a suitable set $\mathcal{L}$ of rational primes of Dirichlet density 1 with certain properties. Then for each $\ell \in \mathcal{L}$ such that η_ℓ is strongly irreducible, we study properties of the reduction $\bar{\eta}_\ell$ and its symmetric square $\mathrm{Sym}^2\bar{\eta}_\ell$, which enable us to conclude Theorem 3.1 by applying Theorem 2.2. Unless specified otherwise, the representations are over algebraically closed fields for convenience. Hence there is no distinction between irreducibility and absolute irreducibility.

The set $\mathcal{L}$ arises from applying results in §5.2 of [BGGT] for a number field F (not necessarily totally real) as follows. Let $\{\mathbb{Q}, S, \{Q_\mathfrak{p}(X)\}, \{r_\ell\}, \{\mathrm{v}_\tau\}\}$ be a strongly compatible system of representations of G_F as in Definition 2. Assume that, after conjugating by some $g_\ell \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, $r_\ell(G_F) \subset \mathrm{GL}_n(\mathbb{Q}_\ell)$ for all ℓ.[2] Let V_ℓ denote the $\mathbb{Q}_\ell$-ambient space of r_ℓ with dimension n. Let G_ℓ be the Zariski closure

[2]Note that $r_\ell(G_F) \subset \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ by the definition of a compatible system. It is not clear that in general we can find $g_\ell \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ so that $r_\ell(G_F) \subset \mathrm{GL}_n(\mathbb{Q}_\ell)$ after conjugating by g_ℓ, although the characteristic polynomial of each $h \in G_\mathbb{Q}$ has coefficients in $\mathbb{Q}_\ell$. Luckily this is true for $r_\ell = \rho_\ell$ or $r_\ell = \rho_\ell \otimes \rho_\ell$ used below.

of $r_\ell(G_F)$ inside $\mathrm{GL}_n(\mathbb{Q}_\ell)$ with the identity component G_ℓ°, G_ℓ^{ad} the quotient of G_ℓ° by its radical, and G_ℓ^{sc} the simply connected cover of G_ℓ^{ad}. Then we get maps

$$G_\ell^\circ \overset{\sigma}{\twoheadrightarrow} G_\ell^{\mathrm{ad}} \overset{\tau}{\twoheadleftarrow} G_\ell^{\mathrm{sc}}\ . \tag{3.1}$$

As in the beginning of §5.2 in [BGGT], let Z_ℓ denote the center of G_ℓ° and $H_\ell := G_\ell^{\mathrm{sc}} \times Z_\ell$. Then there is a natural surjection of algebraic groups $H_\ell \twoheadrightarrow G_\ell^\circ$ with a finite and central kernel. So the G_F-action on the ambient space V_ℓ of r_ℓ induces a representation of G_ℓ^{sc} on V_ℓ. In particular, if σ_ℓ is a subrepresentation of $\overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}_\ell} r_\ell$, then the G_ℓ^{sc}-action on V_ℓ leaves invariant the ambient space of σ_ℓ (contained in $\overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}_\ell} V_\ell$). Now apply Proposition 5.2.2 of [BGGT] (where no regularity condition, i.e., Hodge-Tate weights being distinct, is required) to our compatible system $\{r_\ell = \rho_\ell \otimes_{\mathbb{Q}_\ell} \rho_\ell\}$ restricted to G_K. Then we see that G_ℓ^{sc} acts on the ambient space W_ℓ of $\mathrm{Sym}^2\eta_\ell$. Furthermore, Proposition 5.2.2 of [BGGT] shows that there exists a subset $\mathcal{L}$ of rational primes of Dirichlet density 1 with the following properties: for any $\ell \in \mathcal{L}$, there exists a semi-simple group scheme $\widetilde{G}_\ell^{\mathrm{sc}}$ over $\mathbb{Z}_\ell$ with generic fiber G_ℓ^{sc} so that $\widetilde{G}_\ell^{\mathrm{sc}}(\mathbb{Z}_\ell) = \tau^{-1}(\sigma(r_\ell(G_F) \cap G_\ell^\circ))$, where σ and τ are maps described in (3.1). Also there exists a $\mathbb{Z}_\ell$-lattice Λ inside V_ℓ so that the actions of G_ℓ^{sc} and G_K on V_ℓ can be naturally extended to $\widetilde{G}_\ell^{\mathrm{sc}}$- and G_K-actions on Λ.

Removing finitely many primes from $\mathcal{L}$ if necessary, we further require that each $\ell \in \mathcal{L}$ satisfies the following three conditions:

(i) K is unramified above ℓ,

(ii) $\ell - 1 > 6\kappa$,

(iii) $\rho_\ell|_{G_{\mathbb{Q}_\ell}}$ is crystalline. This follows from the fact that ρ_ℓ forms a strongly compatible system as proved by Scholl.

Next we describe some properties of strongly irreducible η_ℓ with $\ell \in \mathcal{L}$.

Lemma 3.4. *Let $\ell \in \mathcal{L}$. Write U_ℓ for the ambient space of η_ℓ in Theorem 3.1. Suppose that η_ℓ is strongly irreducible. Then the G_ℓ^{sc}-action on U_ℓ factors through the action of SL_2. In particular, the G_ℓ^{sc} -action on the ambient space W_ℓ of $\mathrm{Sym}^2\eta_\ell$ is irreducible.*

Proof. Denote by N the identity component of the Zariski closure of $\eta_\ell(G_K)$ in $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ which acts on U_ℓ. The projective image of N in PGL_2 is a connected subgroup, which we claim to be the whole group. For this, it suffices to look at the Lie algebra $\mathrm{Lie}(\mathrm{PGL}_2)$ by Proposition 3.22 of Milne's note [Mil]. Since $\mathrm{Lie}(\mathrm{PGL}_2)$ consists of 2×2 matrices with trace 0, one finds that the nontrivial connected algebraic subgroups of PGL_2, up to conjugation, have four possibilities: the split torus, the unipotent subgroup, the Borel subgroup, and PGL_2 itself. Since the Zariski closure of $\eta_\ell(G_K)$ has finitely many connected components, the first three cases imply the reducibility of $\eta_\ell|_{G_M}$ for some finite extension M over K, contradicting the assumption on η_ℓ being strongly irreducible. So the projective image of N is PGL_2. It is easy to check that SL_2 is contained in the commutator subgroup $[\mathrm{PGL}_2, \mathrm{PGL}_2] = [N, N]$ of N. Hence $\mathrm{SL}_2 \subset N$. Finally, since $\det(\eta_\ell) = \chi\epsilon_\ell^{1-\kappa}$ with χ being a finite character by Proposition 3.2 and $\kappa \geq 2$, so N/SL_2 must have dimension 1. This shows that N has dimension 4. Since N is connected, N must be GL_2. This shows that the G_ℓ^{sc}-action on U_ℓ factors through the action of $\mathrm{GL}_2^{\mathrm{sc}} = \mathrm{SL}_2$, and then the G_ℓ^{sc}-action on $W_\ell = \mathrm{Sym}^2 U_\ell$ is irreducible. ☐

Proposition 3.5. *Let $\ell \in \mathcal{L}$. If η_ℓ in Theorem* 3.1 *is strongly irreducible, then the reduction of* $\mathrm{Sym}^2\eta_\ell|_{G_{K(\zeta_\ell)}}$ *is irreducible.*

Proof. We first remark that Proposition 5.3.2 in [BGGT] cannot be applied directly because $\rho_\ell \otimes \rho_\ell$ may not be regular. Fortunately, we can follow their idea, which is built upon the work [Lar] of Larsen.

By Lemma 3.4, if $\eta_\ell|_{G_M}$ is strongly irreducible, then $\mathrm{Sym}^2\eta_\ell$ is an absolutely irreducible G_ℓ^{sc}-module. Then, by Proposition 5.3.2 (6) of [BGGT], there exists a finite unramified extension M_λ of $\mathbb{Q}_\ell$ so that $\mathrm{Sym}^2\eta_\ell$ is defined over M_λ as an absolutely irreducible $\widetilde{G}_\ell^{\mathrm{sc}}$-module. Finally, the mod λ reduction of $(\mathcal{O}_{M_\lambda} \otimes_{\mathbb{Z}_\ell} \Lambda) \cap \mathrm{Sym}^2\eta_\ell$ is absolutely irreducible as a $\widetilde{G}_\ell^{\mathrm{sc}}(\mathbb{Z}_\ell)$-module. Therefore, we conclude that as a G_K-module, the reduction of $\mathrm{Sym}^2\eta_\ell$ is absolutely irreducible.

It remains to prove that $\mathrm{Sym}^2\bar{\eta}_\ell|_{G_{K(\zeta_\ell)}}$ is irreducible. By construction, K is unramified over ℓ so that $\mathrm{Gal}(K(\zeta_\ell)/K) \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times$. Write $\overline{W}_\ell$ for the ambient space of $\mathrm{Sym}^2\bar{\eta}_\ell$. Suppose $\overline{W}_\ell|_{G_{K(\zeta_\ell)}}$ is reducible and we will derive a contradiction. Since $\overline{W}_\ell$ is irreducible, by Clifford theory recalled in §2.6, we see that $\overline{W}_\ell|_{G_{K(\zeta_\ell)}}$ is a direct sum of characters χ_i of $G_{K(\zeta_\ell)}$. Let $\chi = \chi_1$ and set $H := \{g \in G_K | \chi^g \simeq \chi\}$ and $M = (\overline{\mathbb{Q}})^H \subset K(\zeta_\ell)$. By Theorem 2.3, $\overline{W}_\ell \simeq \mathrm{Ind}_{G_M}^{G_K}(\chi' \otimes \gamma)$ for a character χ' of G_M extending χ and a representation γ of G_M with finite image. Since $\overline{W}_\ell$ has dimension 3, either $M = K$ or $[M:K] = 3$.

Case 1: $M = K$. In this case, we have $\overline{W}_\ell \simeq \chi' \otimes \gamma$ where χ' is a character of G_K extending χ. Then $\overline{W}'_\ell := ((\chi')^{-1} \otimes \overline{W}_\ell)|_{G_{K(\zeta_\ell)}}$ is trivial. So $\overline{W}'_\ell$ is a 3-dimensional representation of the cyclic group $\mathrm{Gal}(K(\zeta_\ell)/K)$ and hence must be reducible, so is $\overline{W}_\ell$, a contradiction.

Case 2: $[M:K] = 3$. In this case, χ can be extended to a character of G_M so that $\overline{W}_\ell \simeq \mathrm{Ind}_{G_M}^{G_K}\chi$ and M is the unique subfield of $K(\zeta_\ell)$ with degree-3 over K. We have $\ell - 1 > \kappa$. Let v be a prime of K above ℓ with $[K_v : \mathbb{Q}_\ell] = f$ and let I_v be the inertia subgroup of $G_v := \mathrm{Gal}(\overline{K}_v/K_v)$. It follows from the Fontaine-Lafaillle theory that $\bar{\eta}_\ell|_{I_v} \simeq \omega_{2f}^h \oplus \omega_{2f}^{h\ell}$ (resp., $(\bar{\eta}_\ell|_{I_v})^{\mathrm{ss}} = \omega_f^a \oplus \omega_f^b$) if $\bar{\eta}_\ell|_{G_v}$ is irreducible (resp., reducible). Here ω_m denotes the fundamental character given by

$$\omega_m(g) = \frac{g(\sqrt[\ell^m-1]{\ell})}{\sqrt[\ell^m-1]{\ell}} \tag{3.2}$$

for $g \in G_K$, and $h = \sum_{i=0}^{2f-1} h_i p^i$ with $\{h_i, h_{i+f}\} = \{0, 1-\kappa\}$ (resp., $a = \sum_{i=1}^{f-1} a_i p^i$ and $b = \sum_{i=0}^{f-1} b_i p^i$ with $\{a_i, b_i\} = \{0, 1-\kappa\}$).

Consequently, $\overline{W}_\ell|_{I_v} \simeq \omega_{2f}^{2h} \oplus \omega_{2f}^{h(1+\ell)} \oplus \omega_{2f}^{2h\ell}$ or $(\overline{W}_\ell|_{I_v})^{\mathrm{ss}} \simeq \omega_f^{2a} \oplus \omega_f^{a+b} \oplus \omega_f^{2b}$. Since $K(\zeta_\ell)$ is totally ramified at v, so is M. Denote by v' the only place of M above v. Then the inertia subgroup $I_{v'}$ of $\mathrm{Gal}(\overline{K}_v/M_{v'})$ is an index-3 subgroup of I_v. Let τ be a generator of $\mathrm{Gal}(M/K)$. We have $\overline{W}_\ell|_{G_M} \simeq \chi \oplus \chi^\tau \oplus \chi^{\tau^2}$, and similarly for $\overline{W}_\ell|_{I_{v'}}$.

Therefore the set $\{\chi, \chi^\tau, \chi^{\tau^2}\}$ must match with either $\{\omega_{2f}^{2h}, \omega_{2f}^{h(1+\ell)}, \omega_{2f}^{2h\ell}\}$ or $\{\omega_f^{2a}, \omega_f^{a+b}, \omega_f^{2b}\}$ when restricted to $I_{v'}$. Since the fundamental characters all factor through the tame inertia subgroup, which is commutative, it is easy to check that the restrictions of χ, χ^τ, and χ^{τ^2} to $I_{v'}$ are isomorphic and hence identical. To derive a contradiction, it suffices to show that $\omega_{2f}^{2h}, \omega_{2f}^{h(1+\ell)}, \omega_{2f}^{2h\ell}$ restricted to $I_{v'}$

are distinct, and so are the restrictions of ω^{2a}, ω_f^{a+b}, ω_f^{2b} to $I_{v'}$. By (3.2), the image of ω_{2f} is a cyclic group of order $\ell^{2f}-1$. Since $I_{v'}$ is a subgroup of I_v with index-3, so $\omega_{2f}(I_{v'})$ is a cyclic group with order at least $(\ell^{2f}-1)/3$. Since $\ell-1>6\kappa$, we see that $6h$, $3h(1+\ell)$, $6h\ell$ are distinct inside $\mathbb{Z}/(\ell^{2f}-1)\mathbb{Z}$. Then $\omega_{2f}^{2h}, \omega_{2f}^{h(1+\ell)}, \omega_{2f}^{2h\ell}$ restricted to $I_{v'}$ are all distinct. The same conclusion can be drawn for the triple ω_f^{2a}, ω_f^{a+b}, ω_f^{2b}. $\square$

With the above preparation, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\ell \in \mathcal{L}$. We distinguish two cases.

(I). The representation η_ℓ is strongly irreducible. By Proposition 3.5, $\mathrm{Sym}^2\bar{\eta}_\ell|_{G_{K(\zeta_\ell)}}$ is irreducible, and hence so is $\mathrm{Sym}^2\bar{\eta}_\ell|_{G_{F(\zeta_\ell)}}$. Then Theorem 3.1 follows from Theorem 2.2.

(II). The representation η is potentially reducible. Then η_ℓ is totally odd and automorphic by Proposition 3.3. $\square$

3.4. **Applications of Theorem 3.1.** More about the representations η_ℓ in Theorem 3.1 can be concluded provided that we have more information on the characteristic polynomials at the Frobenius elements in G_K, as shown below.

Proposition 3.6. *Suppose that the representations η_ℓ in Theorem* 3.1 *satisfy the additional condition*

(C) *There is a finite set of primes S of K so that at each prime $\mathfrak{p}$ of K outside S, the characteristic polynomial of $\eta_\ell|_{G_K}(\mathrm{Frob}_\mathfrak{p})$ is independent of the primes ℓ not divisible by $\mathfrak{p}$.*

Then $\{\eta_\ell|_{G_K}\}$ forms a compatible system. If we further assume that K/F is a solvable extension, then η_ℓ is potentially automorphic for all ℓ.

Proof. To prove that $\{\eta_\ell|_{G_K}\}$ forms a compatible system, it suffices to show the existence of a number field containing coefficients of the characteristic polynomials of $\eta_\ell|_{G_K}(\mathrm{Frob}_\mathfrak{p})$ for all primes $\mathfrak{p}$ of K not in S.

We first assume that there exists a prime ℓ' so that $\eta_{\ell'}$ is potentially reducible. Then Proposition 3.3 implies that $\eta_{\ell'}$ is automorphic. In particular, there exists a compatible system of *automorphic* ℓ-adic Galois representations $\{E, S_1, \{Q_{\tilde{\mathfrak{p}}}(X)\}, \{\tilde{\eta}_\lambda\}\}$ of G_F so that $\eta_{\ell'} \simeq \tilde{\eta}_\lambda$ for a prime λ of E above ℓ'. After restricting the representations to G_K, we obtain a compatible system $\{E, S_2, \{Q_\mathfrak{p}(X)\}, \{\tilde{\eta}_\lambda|_{G_K}\}\}$ of representations of G_K. Here S_1 (resp., S_2) is a finite set of places of F (resp., K), and $\tilde{\mathfrak{p}}$ (resp., $\mathfrak{p}$) runs through all finite places of F (resp., K) outside S_1 (resp., S_2), and E contains the coefficients of the characteristic polynomials $Q_{\tilde{\mathfrak{p}}}(X)$ and $Q_\mathfrak{p}(X)$. In view of condition (C), for almost all primes $\mathfrak{p}$, $Q_\mathfrak{p}(X)$ is the characteristic polynomial of $\eta_\ell|_{G_K}(\mathrm{Frob}_\mathfrak{p})$ for $\ell=\ell'$ and hence all primes ℓ not divisible by $\mathfrak{p}$. Consequently $\{E, S_2, \{Q_\mathfrak{p}(X)\}, \{\eta_\lambda|_{G_K}\}\}$ forms a compatible system, where $\eta_\lambda=\eta_\ell$ for all primes λ of E dividing ℓ. If K/F is solvable, then by the solvable base change theorem in [AC], we see that $\{\tilde{\eta}_\lambda|_{G_K}\}$ is automorphic, so are $\eta_\ell|_{G_K}$.

Next we assume that for each prime ℓ, η_ℓ is strongly irreducible. Let $\mathcal{L}$ be the set of rational primes as in Theorem 3.1. Choose a prime $\ell' \in \mathcal{L}$. Then Proposition 3.5 and Theorem 5.5.1 in [BGGT] imply the existence of a compatible system of ℓ-adic Galois representations $\{E, \tilde{S}, \{Q_{\tilde{\mathfrak{p}}}(X)\}, \{\tilde{\eta}_\lambda\}\}$ of G_F so that $\eta_{\ell'} \simeq \tilde{\eta}_\lambda$ for a prime λ of E above ℓ'. So by restricting $\{\tilde{\eta}_\lambda\}$ to G_K, the same argument as the potentially reducible case above shows that $\{\eta_\ell|_{G_K}\}$ forms a compatible system.

Furthermore, by Theorem 3.1 there exists a finite extension L/F so that $\eta_{\ell'}|_{G_L}$ is automorphic. If K is solvable over F, then so is KL/L. Hence by solvable base change, $\eta_{\ell'}|_{G_{LK}}$ is automorphic, corresponding to an automorphic representation π of GL_2 over KL. Now using the facts that the system $\{\eta_\ell|_{G_{KL}}\}$ is compatible, all $\eta_\ell|_{G_{KL}}$ are irreducible and they are determined by the traces of the elements in G_{KL}, we conclude that every $\eta_\ell|_{G_{KL}}$ is automorphic, corresponding to the same representation π as $\eta_{\ell'}|_{G_{KL}}$. This proves that η_ℓ is potentially automorphic for all primes ℓ. $\square$

Next we draw some consequences on (potential) automorphy of Scholl representations from Theorem 3.1.

Proposition 3.7. *Let $\{\rho_\ell\}$ be a compatible system of Scholl representations of $G_{\mathbb{Q}}$. Let K be a finite solvable extension of a totally real field F. Suppose that for each ℓ we have*

1. *$\rho_\ell|_{G_K} \simeq \bigoplus_{i=1}^d \sigma_{\ell,i}$ where $\sigma_{\ell,i}$ are degree-2 representations of G_K;*
2. *for each $1 \le i \le d$ there is a 2-dimensional representation $\eta_{\ell,i}$ of G_F such that $\eta_{\ell,i}|_{G_K}$ is finitely projectively equivalent to $\sigma_{\ell,i}$.*

Then ρ_ℓ is potentially automorphic for all ℓ.

Notice that the representations $\eta_{\ell,i}$ in (2) above are by no means unique. For instance, one may twist $\eta_{\ell,i}$ by finite characters of G_F. Indeed, the following lemma assures us that by so doing we may assume that the representations $\eta_{\ell,i}$ in (2) are crystalline at each prime v of F above ℓ as long as $\rho_\ell|_{G_{\mathbb{Q}_\ell}}$ is crystalline.

Lemma 3.8. *For each $\eta_{\ell,i}$ in Proposition 3.7 with ℓ large enough so that $\rho_\ell|_{G_{\mathbb{Q}_\ell}}$ is crystalline, there exists a finite character $\chi_{\ell,i}$ of G_F so that $\chi_{\ell,i} \otimes \eta_{\ell,i}|_{G_{F_v}}$ is crystalline at each prime v of F above ℓ.*

Proof. Denote by V_1 the ambient space of $\eta_{\ell,i}$ and V_2 its dual. Then $(V_1 \otimes V_2)|_{G_K} \simeq \sigma_{\ell,i} \otimes (\sigma_{\ell,i})^\vee$ is crystalline at all primes of K above large ℓ. Now applying Proposition 3.3.4 in [LY] to $V_1 \otimes V_2$, we see the existence of a character $\chi_{\ell,i}$ of G_F with finite image so that $\chi_{\ell,i} \otimes \eta_{\ell,i}$ is crystalline at all primes v of F above ℓ. $\square$

Now we proceed to prove Proposition 3.7.

Proof. We follow the same idea as the proof of Theorem 3.1. Note first that the set $\mathcal{L}$ of primes constructed in the previous subsection is independent of the decomposition of ρ_ℓ. Choose a large $\ell \in \mathcal{L}$ so that Lemma 3.8 holds. Then we may assume that $\eta_{\ell,i}$ in (2) are crystalline at the primes v of F above ℓ. Let J_ℓ be the set of i's such that $\eta_{\ell,i}$ is strongly irreducible. Then for $i \in J_\ell$ we know that $\mathrm{Sym}^2 \bar{\eta}_{\ell,i}|_{G_{F(\zeta_\ell)}}$ is irreducible by Proposition 3.5. Hence Theorem 2.2 (2) applies to $\eta_{\ell,i}$ for $i \in J_\ell$ and we obtain a finite totally real Galois extension $F' = F'(\ell)$ over F such that $\eta_{\ell,i}|_{G_{F'}}$ is automorphic for $i \in J_\ell$. For $i \notin J_\ell$, we see that $\eta_{\ell,i}$ is induced from a character of the Galois group of a quadratic CM extension of F by Proposition 3.3. It is clear that $\eta_{\ell,i}|_{G_{F'}}$ is also an induction of a character of the Galois group of a quadratic CM extension of F'. Thus $\eta_{\ell,i}|_{G_{F'}}$ is automorphic. Since $\eta_{\ell,i}|_{G_K}$ is finitely projectively equivalent to $\sigma_{\ell,i}$, there is a character χ_i of G_K so that $\eta_{\ell,i}|_{G_K} \simeq \chi_i \otimes \sigma_{\ell,i}$ for each i. In particular, there exists a finite abelian extension K_i/K so that $\eta_{\ell,i}|_{G_{K_i}} \simeq \sigma_{\ell,i}|_{G_{K_i}}$. Let K' be the composite of all K_i and F'. Then we see that K'/F' is a finite solvable extension. Thus we conclude that

$\sigma_{\ell,i}|_{G_{K'}} = \eta_{\ell,i}|_{G_{K'}}$ is automorphic. This proves that $\rho_\ell|_{G_{K'}}$ is automorphic for one and hence all ℓ as ρ_ℓ is a compatible system. $\square$

An immediate consequence of Proposition 3.7 and Theorem 3.1 is the following.

Corollary 3.9. *Let $\{\rho_\ell\}$ be a compatible system of Scholl representations of $G_{\mathbb{Q}}$. Suppose there exists a totally real field F so that for each ℓ we have $\rho_\ell|_{G_F} \simeq \bigoplus_{i=1}^d \eta_{\ell,i}$ with $\eta_{\ell,i}$ degree-2 representations of G_F. Then ρ_ℓ is potentially automorphic for all ℓ. Moreover, if $F = \mathbb{Q}$, then all ρ_ℓ are automorphic.*

4. Potentially GL_1- or GL_2-type Scholl representations

Throughout this section, $\{\rho_\ell\}$ denotes a system of compatible $2d$-dimensional Scholl representations of $G_{\mathbb{Q}}$ associated to a d-dimensional space of weight κ cusp forms of a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We assume that *$\rho_\ell$ are absolutely irreducible for all ℓ* in this section. By Theorem 2.3 and Lemma 5.3.1 of [BGGT] (1), there is a finite Galois extension $L/\mathbb{Q}$ such that, for each ℓ, the restriction $\rho_\ell|_{G_L}$ decomposes as

$$\rho_\ell|_{G_L} \simeq \sigma_{\ell,1} \oplus \cdots \oplus \sigma_{\ell,m(\ell)}, \tag{4.1}$$

where $\sigma_{\ell,i}$ are strongly irreducible and conjugate to each other under $\mathrm{Gal}(L/\mathbb{Q})$, hence they are of the same dimension.

Remark 1. We conjecture that, with L fixed, the number $m(\ell)$ and hence $\dim_{\overline{\mathbb{Q}}_\ell} \sigma_{\ell,i}$ are independent of ℓ.

In the remainder of this section, we assume that $\dim_{\overline{\mathbb{Q}}_\ell} \sigma_{\ell,1}$ is independent of ℓ, and discuss the case $\dim_{\overline{\mathbb{Q}}_\ell} \sigma_{\ell,1} = 1$ or 2.

4.1. Scholl representations of GL_1-type.

Consider first the case that $\dim_{\overline{\mathbb{Q}}_\ell} \sigma_{\ell,1} = 1$. By Theorem 2.3, $\rho_\ell \simeq \mathrm{Ind}_{G_M}^{G_{\mathbb{Q}}}(\eta_\ell \otimes \gamma_\ell)$ for some subfield $M := M(\ell)$ of L depending on ℓ a priori. Here η_ℓ is 1-dimensional and γ_ℓ is a representation with finite image.

Let N be the splitting field of γ_ℓ. Since η_ℓ is geometric, it is automorphic. Hence $\rho_\ell|_{G_N}$ is automorphic and then ρ_ℓ is potentially automorphic. This proves the following.

Theorem 4.1. *Suppose that the Scholl representation ρ_ℓ is potentially of* GL_1*-type for one prime ℓ. Then ρ_ℓ is potentially automorphic for all ℓ.*

In general, it is difficult to prove that ρ_ℓ is automorphic without further information on M and γ_ℓ.

Remark 2. The maximal CM subfield M_0 of M cannot be totally real. In particular, M cannot be $\mathbb{Q}$. Indeed, suppose that M_0 is totally real. Then we see from Lemma 2.1 (3) that $\mathrm{HT}_g(\eta_\ell)$ is independent of $g \in \mathrm{Hom}_{\mathbb{Q}}(M, \overline{\mathbb{Q}}_\ell)$. Consequently $\mathrm{HT}_\tau(\sigma_{\ell,1})$ is independent of $\tau \in \mathrm{Hom}_{\mathbb{Q}}(L, \overline{\mathbb{Q}}_\ell)$ (note that $L/\mathbb{Q}$ is Galois) and so is $\mathrm{HT}_\tau(\sigma_{\ell,1}^g)$ by Lemma 2.1 (2). Then $\rho_\ell|_{G_L} \simeq \sigma_{\ell,1} \oplus \cdots \oplus \sigma_{\ell,m(\ell)}$ has only one Hodge-Tate weight independent of $\tau \in \mathrm{Hom}_{\mathbb{Q}}(L, \overline{\mathbb{Q}}_\ell)$, contradicting the fact that ρ_ℓ has two distinct Hodge-Tate weights, 0 and $-\kappa + 1$.

4.2. **Scholl representations of GL_2-type.** Next we consider the situation that $\dim_{\overline{\mathbb{Q}}_\ell} \sigma_{\ell,1} = 2$ for all ℓ. Combining Theorem 2.3 and Proposition 3.2, we have the following result.

Proposition 4.2. *Under the notation and assumptions of this section, for each ℓ there exist a subfield $M = M(\ell)$ of L and Galois representations $\gamma_\ell : G_M \to \mathrm{GL}_d(\overline{\mathbb{Q}}_\ell)$ and $\eta_\ell : G_M \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that*

(1) *γ_ℓ has finite image and $\eta_\ell|_{G_L}$ is projectively equivalent to $\sigma_{\ell,1}$;*
(2) *$\rho_\ell \simeq \mathrm{Ind}_{G_M}^{G_\mathbb{Q}}(\gamma_\ell \otimes \eta_\ell)$;*
(3) *if M is totally real, then for each embedding $\tau : M \hookrightarrow \overline{\mathbb{Q}}_\ell$ we have $\mathrm{HT}_\tau(\eta_\ell) = \{0, 1-\kappa\}$.*

Remark 3. The representations γ_ℓ and η_ℓ in the above corollary are not unique since they may be, respectively, replaced by $\gamma_\ell \otimes \psi^{-1}$ and $\eta_\ell \otimes \psi$ for any finite character ψ of G_M. As in the proof of Lemma 3.8, Proposition 3.3.4 in [LY] implies that, when $\rho_\ell|_{G_{\mathbb{Q}_\ell}}$ is crystalline, there exists a finite character ψ of G_M so that, after replacing η_ℓ by $\eta_\ell \otimes \psi$, we have $\eta_\ell|_{G_{M_\mathfrak{q}}}$ crystalline at every prime $\mathfrak{q}$ of M dividing ℓ.

Theorem 4.3. *Keep the same assumptions and notation as in this subsection. Assume further that there is a finite set of primes S of L so that at each prime $\mathfrak{p}$ of L outside S, the characteristic polynomial of $\sigma_{\ell,1}(\mathrm{Frob}_\mathfrak{p})$ is independent of the primes ℓ not divisible by $\mathfrak{p}$. Then the following statements hold:*

(1) *The field $M(\ell)$ is independent of ℓ; denote it by M.*
(2) *Assume that M is totally real and L is solvable over M; then ρ_ℓ is potentially automorphic.*

Proof. (1). Write $\sigma_\ell := \sigma_{\ell,1}$, which is strongly irreducible by assumption. We know from Theorem 2.3 that the $\sigma_{\ell,i}$'s are conjugates of σ_ℓ by $G_\mathbb{Q}$. Recall that an irreducible 2-dimensional ℓ-adic Galois representation is determined by its trace; see [Ser1]. Hence for $g \in G_\mathbb{Q}$, $\sigma_\ell \simeq \sigma_\ell^g$ is equivalent to $\mathrm{Tr}(\sigma_\ell) = \mathrm{Tr}(\sigma_\ell^g)$. Let ℓ' be another prime. The assumption that for almost all primes $\mathfrak{p}$ of L the characteristic polynomial of $\sigma_\ell(\mathrm{Frob}_\mathfrak{p})$ is independent of ℓ not divisible by $\mathfrak{p}$ implies that $\mathrm{Tr}(\sigma_\ell^g) = \mathrm{Tr}(\sigma_{\ell'}^g)$ for all $g \in G_\mathbb{Q}$ by Cebotarev density theorem. So $\sigma_\ell \simeq \sigma_\ell^g$ if and only if $\sigma_{\ell'} \simeq \sigma_{\ell'}^g$. Hence $M(\ell)$ is independent of ℓ.

(2). Since $\eta_\ell|_{G_L}$ is projectively equivalent to $\sigma_{\ell,1}$ by Proposition 4.2, we easily see that (2) is a consequence of Proposition 3.7. □

Remark 4. With the assumption of the above theorem and the further assumption that M is totally real, Proposition 3.6 shows that $\{\sigma_\ell\}$ also forms a compatible system.

4.3. **Potentially 2-isotypic case.** More can be said about the automorphy of Scholl representations ρ_ℓ when they are potentially 2-isotypic. This is stated in the theorem below.

Theorem 4.4. *Let $\{\rho_\ell\}$ be a compatible system of $2d$-dimensional semi-simple subrepresentations of Scholl representations of $G_\mathbb{Q}$ which are all 2-isotypic when restricted to G_F for a finite Galois extension $F/\mathbb{Q}$. Suppose that F contains a solvable extension $K/\mathbb{Q}$ such that for each ℓ the representation $\rho_\ell \simeq \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \sigma_\ell$ for a 2-dimensional representation σ_ℓ of G_K. Then all ρ_ℓ are automorphic.*

Proof. Since $\{\rho_\ell\}$ forms a compatible system, it suffices to show that ρ_ℓ is automorphic for one ℓ.

As ρ_ℓ is potentially 2-isotypic over F and $\rho_\ell \simeq \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}\sigma_\ell$ with $K \subset F$, this forces σ_ℓ to be irreducible. Let ρ'_ℓ be an irreducible subrepresentation of ρ_ℓ so that σ_ℓ is a subrepresentation of $\rho'_\ell|_{G_K}$. Since ρ_ℓ is potentially 2-isotypic, so is ρ'_ℓ. By Theorem 2.3, $\rho'_\ell \cong \eta_\ell \otimes \gamma_\ell$ for a 2-dimensional representation η_ℓ and a representation γ_ℓ of $G_\mathbb{Q}$, where $\eta_\ell|_{G_K}$ is finitely projectively equivalent to σ_ℓ and γ_ℓ has finite image. It follows from Proposition 3.2 that η_ℓ is irreducible with two distinct Hodge-Tate weights 0 and $1-\kappa$ for some integer $\kappa \geq 2$. If η_ℓ is potentially reducible, then it is odd and automorphic by Proposition 3.3. So we now assume that η_ℓ is strongly irreducible for all ℓ and show that the same automorphy conclusion holds for some η_ℓ.

By Lemma 3.8, for ℓ large so that $\rho_\ell|_{G_{\mathbb{Q}_\ell}}$ is crystalline, there exists a finite character ξ_ℓ of $G_\mathbb{Q}$ such that $\eta_\ell \otimes \xi_\ell$ is crystalline at ℓ. Replacing η_ℓ by $\eta_\ell \otimes \xi_\ell$ and γ_ℓ by $\xi_\ell^{-1} \otimes \gamma_\ell$ if necessary, we may assume that η_ℓ is crystalline above ℓ. Choose a large ℓ so that it satisfies the conditions for $\mathcal{L}$ in §3.3 with $F = \mathbb{Q}$ and K as in Theorem 4.4. As $\eta_\ell|_{G_K}$ is strongly irreducible, by Proposition 3.5 $\mathrm{Sym}^2\bar{\eta}_\ell|_{G_{K(\zeta_\ell)}}$ is absolutely irreducible, and hence the same holds for $\mathrm{Sym}^2\bar{\eta}_\ell|_{G_{\mathbb{Q}(\zeta_\ell)}}$. Now apply Theorem 2.2(3) to conclude that η_ℓ is automorphic.

Finally since $K/\mathbb{Q}$ is a finite solvable extension, by base change, $\eta_\ell|_{G_K}$ is also automorphic, and so is its finite twist σ_ℓ. Finally, by applying automorphic induction to σ_ℓ over the solvable extension $K/\mathbb{Q}$, we obtain the automorphy of $\rho_\ell \cong \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}\sigma_\ell$, as desired. □

Remark 5. In [AL^3] the authors showed that degree-4 Scholl representations of $G_\mathbb{Q}$ admitting quaternion multiplication are automorphic. As shown in Theorem 3.1.2 of [AL^3], such representations are special cases of the degree-4 representations in Theorem 4.4.

4.4. **Some examples of potentially 2-isotypic representations.**

4.4.1. *Weight 2 examples.* Using Belyi's Theorem, every smooth irreducible projective curve C defined over $\overline{\mathbb{Q}}$ is isomorphic to the modular curve of a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, which is not unique in general. In this regard, the Galois representations on the Jacobian of C, when C is defined over $\mathbb{Q}$, may be viewed as Scholl representations of $G_\mathbb{Q}$ associated to $S_2(\Gamma)$, the space of weight 2 cusp forms of Γ.

Example 1. Consider the family of curves $C_b : y^2 = x^6 + bx^3 + 1$ with generic genus 2. Here we assume $b \in \mathbb{Q}$. On C_b there are two maps, which are $\tau_1 : (x, y) \mapsto (\zeta_3 x, y)$ defined over $\mathbb{Q}(\zeta_3)$, and $\tau_2 : (x, y) \mapsto (\frac{1}{x}, \frac{y}{x^3})$ defined over $\mathbb{Q}$. They are of order 3 and 2, respectively. Together they generate a finite group isomorphic to the dihedral group of order 12. For special choices of b such as $b = \pm 2$, the curve has smaller genus and for other choices of b such as $b = 0$, the curve is a quotient of a Fermat curve and hence the corresponding 4-dimensional Galois representations $\rho_{\ell,b}$ decompose into 1-dimensional representations after suitable restriction. For generic b, using τ_2 we decompose $\rho_{\ell,b} = \sigma_{\ell,b,1} \oplus \sigma_{\ell,b,2}$ into the sum of two degree-2 irreducible representations $\sigma_{\ell,b,i}$, $i = 1, 2$, of $G_\mathbb{Q}$ (since τ_2 is defined over $\mathbb{Q}$) over $\mathbb{Q}_\ell$. Further, for each i, as ℓ varies, the family $\{\sigma_{\ell,b,i}\}$ is compatible. So the characteristic polynomials of $\sigma_{\ell,b,i}(\mathrm{Frob}_p)$ at unramified p have coefficients in $\mathbb{Z}$. Moreover, $\rho_{\ell,b}$

restricted to $G_{\mathbb{Q}(\sqrt{-3})}$ commutes with the operator arising from τ_1. Hence at the primes $p \equiv 1 \mod 3$ splitting in $\mathbb{Q}(\sqrt{-3})$ where $\rho_{\ell,b}$ is unramified, the characteristic polynomial of $\rho_{\ell,b}(\mathrm{Frob}_p)$ is a square. This in turn implies that $\sigma_{\ell,b,1}(\mathrm{Frob}_p)$ and $\sigma_{\ell,b,2}(\mathrm{Frob}_p)$ have the same characteristic polynomials because they are over $\mathbb{Z}$. When $p \equiv 2 \mod 3$, we have $\mathbb{Z}/p\mathbb{Z} = (\mathbb{Z}/p\mathbb{Z})^3$ so that over $\mathbb{Z}/p\mathbb{Z}$, the curve C_b is isomorphic to the genus 0 curve $y^2 = s^2 + bs + 1$. Therefore at almost all such primes p, $\rho_{\ell,b}(\mathrm{Frob}_p)$ has trace 0, which implies that $\sigma_{\ell,b,1}(\mathrm{Frob}_p)$ and $\sigma_{\ell,b,2}(\mathrm{Frob}_p)$ have opposite traces and the same determinants. Combined, this shows that $\sigma_{\ell,b,1}$ and $\sigma_{\ell,b,2}$ differ by a quadratic twist associated to $\mathbb{Q}\left(\sqrt{-3}\right)$. Therefore $\rho_{\ell,b}$ is also potentially 2-isotypic over $\mathbb{Q}\left(\sqrt{-3}\right)$. By Corollary 3.9, $\rho_{\ell,b}$ is automorphic for all ℓ.

Both $\sigma_{\ell,b,1}$ and $\sigma_{\ell,b,2}$ are odd and have Hodge-Tate weights 0 and -1, by the now established Serre's conjecture, both correspond to holomorphic weight 2 cuspidal newforms with coefficients in $\mathbb{Z}$. As such, the modularity theorem further implies that these newforms are associated to elliptic curves over $\mathbb{Q}$ with conductor equal to the level of the corresponding form. A well-known bound on the conductor of an elliptic curve over $\mathbb{Q}$ then bounds the possible levels, namely the p-exponent of the level is at most 8 for $p = 2$, 5 for $p = 3$, and 2 for $p \geq 5$. For example, for $b = 1$, the representation $\rho_{\ell,1}$ and hence $\sigma_{\ell,1,1}$ and $\sigma_{\ell,1,2}$ are unramified outside $2, 3, \ell$. Thus the level of the newform f corresponding to the family $\{\sigma_{\ell,1,1}\}$ divides $2^8 \cdot 3^5$. Using the information on the characteristic polynomials of $\rho_{\ell,1}$ at primes $p = 5, 7, 11, 13$ computed by `Magma` and a search among all weight 2 Hecke eigenforms with levels dividing $2^8 \cdot 3^5$ and trivial character, we conclude that f is the weight 2 level 324 non-CM newform labelled by 324.2.1.a in [LMFDB], and that corresponding to $\sigma_{\ell,1,2}$ is the twist of f by the quadratic character associated to $\mathbb{Q}(\sqrt{-3})$.

Example 2. In [DFLST], the authors used hypergeometric functions over finite fields to study Galois representations arising from hypergeometric abelian varieties. In particular, they considered the family of smooth curves $X_t^{[6;4,3,1]}$ obtained from desingularization of the generalized Legendre curves

$$y^6 = x^4(1-x)^3(1-tx)$$

with $t \in \mathbb{Q}\setminus\{0,1\}$. The genus of $X_t^{[6;4,3,1]}$ is 3. It is shown that the Jacobian variety of $X_t^{[6;4,3,1]}$ has a 2-dimensional primitive part J_t^{prim} defined over $\mathbb{Q}$, obtained from removing the subabelian varieties isogenous to factors of the Jacobian varieties obtained from $y^d = x^4(1-x)^3(1-tx)$ where $1 < d < 6, d \mid 6$. The abelian variety J_t^{prim} gives rise to a compatible system of ℓ-adic representations $\rho_{\ell,t} : G_{\mathbb{Q}} \to \mathrm{GL}_4(\mathbb{Q}_\ell)$. Due to the map $(x,y) \mapsto (x, \zeta_6 y)$ on $X_t^{[6;4,3,1]}$ defined over $K = \mathbb{Q}(\zeta_3)$,

$$\rho_{\ell,t}|_{G_K} = \sigma_{\ell,t} \oplus \sigma_{\ell,t}^\tau,$$

where τ is the complex conjugation in $\mathrm{Gal}(K/\mathbb{Q})$. By Example 3 of [DFLST], there is a finite character ψ of G_K trivial on G_{L_t} such that $\sigma_{\ell,t} \cong \sigma_{\ell,t}^\tau \otimes \psi$. Here $L_t = K\left(\sqrt[6]{t\frac{(1-t)^2}{2^4}}\right)$ is a finite Galois extension of $\mathbb{Q}$. Thus $\rho_{\ell,t}$ is 2-isotypic over L_t. For any $t \in \mathbb{Q} \smallsetminus \{0,1\}$ such that $L_t \neq K$ and $\sigma_{\ell,t}$ is strongly irreducible, $\sigma_{\ell,t}$ is not isomorphic to $\sigma_{\ell,t}^\tau$. Hence $\rho_{\ell,t} = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \sigma_{\ell,t}$. For those values of t, we have $\rho_{\ell,t}$ automorphic for all ℓ by Theorem 4.4.

Similarly, when one considers the smooth model $X_t^{[12;9,5,1]}$ of

$$y^{12} = x^9(1-x)^5(1-tx),$$

for generic $t \in \mathbb{Q} \setminus \{0,1\}$, the primitive part of its Jacobian variety is 4-dimensional and the corresponding 8-dimensional Galois representation $\rho_{\ell,t}$ is potentially of GL_2-type such that its restriction to G_K with $K = \mathbb{Q}(\zeta_{12})$ decomposes as

$$\rho_{\ell,t}|_{G_K} = \sigma_{\ell,t} \oplus \sigma_{\ell,t}^{\tau_1} \oplus \sigma_{\ell,t}^{\tau_2} \oplus \sigma_{\ell,t}^{\tau_1\tau_2},$$

where $\tau_1 : \zeta_{12} \mapsto \zeta_{12}^{-1}$ and $\tau_1 : \zeta_{12} \mapsto \zeta_{12}^5$ are in $\mathrm{Gal}(K/\mathbb{Q})$ and $\sigma_{\ell,t}$ is strongly irreducible for generic t. By §7.2 of [DFLST], $\sigma_{\ell,t} \cong \sigma_{\ell,t}^{\tau_1} \otimes \chi_1(t)$ and $\sigma_{\ell,t} \cong \sigma_{\ell,t}^{\tau_2} \otimes \chi_2(t)$, where $\chi_1(t), \chi_2(t)$ are characters of G_K of order dividing 12 whose kernels are the absolute Galois group of $K\left(\sqrt[12]{-27t^2(1-t)^6}\right)$ and $K(\sqrt[6]{t})$, respectively. Thus, for a generic choice of t, none of $\chi_1(t)$, $\chi_2(t)$, $\chi_1(t)\chi_2(t)$ are trivial characters, indicating that the representations $\sigma_{\ell,t}, \sigma_{\ell,t}^{\tau_1}, \sigma_{\ell,t}^{\tau_2}, \sigma_{\ell,t}^{\tau_1\tau_2}$ are pairwise nonisomorphic. Hence $\rho_{\ell,t} = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \sigma_{\ell,t}$. On the other hand, $\rho_{\ell,t}$ is potentially 2-isotypic over the Galois extension $L_t = \mathbb{Q}\left(\zeta_{12}, \sqrt[6]{t}, \sqrt[12]{-27(1-t)^6}\right)$ of $\mathbb{Q}$. Therefore, by Theorem 4.4, $\rho_{\ell,t}$ is automorphic for a generic t.

4.4.2. *A weight* 4 *example.* The following example was originally investigated by the late Oliver Atkin among his unpublished notes. The construction is similar to the cases discussed in [LLY, HLV] except that the space of modular forms in consideration has weight 4 instead of weight 3. The group $\Gamma_0(4)$ has genus 0, 3 cusps, and admitting the Atkin-Lehner involution W_4. A Hauptmodul for $\Gamma_0(4)$ is $t := t(z) = \eta(z)^8/\eta(4z)^8$, where $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), q = e^{2\pi i z}$ denotes the Dedekind eta function. The function t has a simple pole at the cusp infinity and vanishes at the cusp 0. The function $E := \eta(2z)^{16}/\eta(z)^8$ is a weight 4 Eisenstein series for $\Gamma_0(4)$ which vanishes at all cusps of $\Gamma_0(4)$ but 0. The function $t_3(z) := \sqrt[3]{\eta(z)^8/\eta(4z)^8} = \sqrt[3]{t}$ is a Hauptmodul of an index-3 noncongruence subgroup, denoted by Γ, of $\Gamma_0(4)$. One way to see that $t_3(z)$ is a noncongruence modular function is that the Fourier coefficients of its q-expansion have unbounded powers of 3 in the denominators. The modular curve for Γ has a model defined over $\mathbb{Q}$ and it is a three-fold cover of the modular curve for $\Gamma_0(4)$, ramified only at the cusps 0 and ∞ with ramification degree-3. The space $S_4(\Gamma)$ of weight 4 cusp forms is spanned by

$$f_1 = E \cdot (t(z))^{2/3} = \sqrt[3]{\frac{\eta(2z)^{48}}{\eta(z)^8\eta(4z)^{16}}} \quad \text{and} \quad f_2 = E \cdot (t(z))^{1/3} = \sqrt[3]{\frac{\eta(2z)^{48}}{\eta(z)^{16}\eta(4z)^{8}}}.$$

Their Fourier expansions $f_i = \sum_{n\geq 1} a_i(n)q^{n/3}$, $i = 1, 2$, have coefficients in $\mathbb{Q}$.

Let ρ_ℓ be the ℓ-adic 4-dimensional Scholl representation of $G_{\mathbb{Q}}$ attached to $S_4(\Gamma)$. The matrices $\zeta = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $W_4 = \left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right)$ normalize Γ and thus act on forms on Γ via the stroke operator as follows:

$$f_1|_\zeta = {\zeta_3}^2 f_1, \quad f_2|_\zeta = \zeta_3 f_2, \quad f_1|_{W_4} = 2^{16/3} f_2, \quad f_2|_{W_4} = 2^{8/3} f_1.$$

They induce the corresponding actions ζ^* and W_4^* on ρ_ℓ. The operator ζ^* is defined over $K = \mathbb{Q}\left(\sqrt{-3}\right)$ and W_4^* is defined over $F = K\left(\sqrt[3]{2}\right)$, the splitting field of $x^3 - 2$. These two operators generate a group isomorphic to S_3, the symmetric group on three letters, which acts on ρ_ℓ and commutes with $\rho_\ell(G_F)$.

Using ζ^*, we decompose $\rho_\ell|_{G_K} = \sigma_\ell \oplus \sigma_\ell^\tau$, where σ_ℓ is a 2-dimensional strongly irreducible representation of G_K over $\mathbb{Q}(\zeta_3)$ and $\tau \in G_\mathbb{Q} \smallsetminus G_K$. Explicit computations yield the following characteristic polynomials of Frobenius elements and their factorization over $\mathbb{Z}[\zeta_3]$. More precisely, we proceed as follows. We first compute many Fourier coefficients $a_i(n)$ of f_i explicitly. As shown by Scholl [Sch1], for $p \geq 5$, the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_p)$ is a degree-4 polynomial $T^4 + A_3(p)T^3 + A_2(p)T^2 + A_1(p)T + A_0(p) \in \mathbb{Z}[T]$ with all roots of absolute value $p^{3/2}$. This sets the range for $A_j(p)$. Moreover, the Atkin and Swinnerton-Dyer congruences hold for f_i, $i = 1, 2$, and integers $r \geq -1$:

$$\begin{aligned} &a_i(p^{r+2}) + A_3(p)a_i(p^{r+1}) \\ &\qquad + A_2(p)a_i(p^r) + A_1(p)a_i(p^{r-1}) + A_0(p)a_i(p^{r-2}) \equiv 0 \mod p^{3+r}. \end{aligned}$$

Here $a_i(p^r) = 0$ if $r < 0$. See [Sch1] for details. For each prime p listed below, we tested for the first few $r \geq 1$ and $i = 1, 2$, from which all coefficients $A_j(p)$ are determined.]2]

p	Char. poly. of $\rho_\ell(\mathrm{Frob}_p)$	Factorization over $\mathbb{Z}[\zeta_3]$
5	$x^4 - 74x^2 + 5^6$	$(x^2 + 18x + 5^3)(x^2 - 18x + 5^3)$
7	$x^4 + 8x^3 - 279x^2 + 2744x + 7^6$	$(x^2 - 8\zeta_3 x + \zeta_3^2 7^3)(x^2 - 8\zeta_3^2 x + \zeta_3 7^3)$
11	$x^4 + 1366x^2 + 11^6$	$(x^2 + 36x + 11^3)(x^2 - 36x + 11^3)$
13	$x^4 - 10x^3 - 2097x^2 - 21970x + 13^6$	$(x^2 + 10\zeta_3 x + \zeta_3^2 13^3)(x^2 + 10\zeta_3^2 x + \zeta_3 13^3)$
17	$x^4 + 9502x^2 + 17^6$	$(x^2 - 18x + 17^3)(x^2 + 18x + 17^3)$
23	$x^4 + 19150x^2 + 23^6$	$(x^2 + 72x + 23^3)(x^2 - 72x + 23^3)$
31	$(x^2 + 16x + 31^3)^2$	$(x^2 + 16x + 31^3)^2$.

From the data above we see that σ_ℓ is not isomorphic to σ_ℓ^τ; thus $\rho_\ell \cong \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \sigma_\ell$. Moreover, $\sigma_\ell|_{G_F} \cong \sigma_\ell^\tau|_{G_F}$ by the action of W_4^*. Hence ρ_ℓ is 2-isotypic over the Galois extension $F = \mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$ of $\mathbb{Q}$. By Theorem 4.4, ρ_ℓ is automorphic. Alternatively, applying Theorem 2.3, we have $\rho_\ell \cong \eta_\ell \otimes \gamma_\ell$, where $\eta_\ell|_{G_K}$ is finitely projectively equivalent to σ_ℓ and γ_ℓ has finite image. In fact, according to the numerical data computed by Atkin, η_ℓ can be chosen to be the Galois representation attached to either one of the following weight 4 level 12 congruence Hecke eigenforms

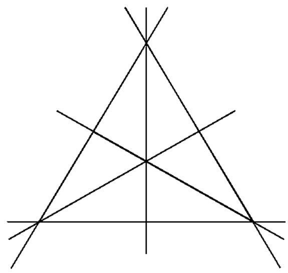
$$g_\pm := g_1(z) \pm 18g_5(z) + 3(g_1(3z) \pm 18g_5(3z))$$

with $g_1(z) = \eta^4(z) \cdot (3E_2(3z) - E_2(z))/2$ and $g_5 = \eta(z)^2\eta(3z)^6$ in which E_2 is the weight 2 nonholomorphic Eisenstein series. Here $g_\pm$ differ by twisting by the quadratic character χ_{-3} corresponding to $\mathbb{Q}(\sqrt{-3})$. The representation γ_ℓ is induced from a finite character of G_K, hence is also automorphic. Therefore ρ_ℓ corresponds to an automorphic representation of $\mathrm{GL}_2 \times \mathrm{GL}_2$ over $\mathbb{Q}$, hence is automorphic by [Ram].

5. An infinite family of potentially automorphic Scholl representations attached to weight 3 noncongruence forms

In this section we apply the results of previous sections to prove the potential automorphy of an infinite family of Scholl representations attached to weight 3 cusp forms for the noncongruence subgroups explicitly constructed in [ALL]. This gives the first family of Scholl representations of $G_\mathbb{Q}$ with unbounded degree for which the potential automorphy is established.

5.1. **A family of elliptic surfaces $\mathcal{E}_n$.** It is well known that there are thirteen $K3$ surfaces defined over $\mathbb{Q}$ whose Néron-Severi group has rank 20, generated by algebraic cycles over $\mathbb{Q}$. Elkies and Schütt [ES] have constructed them from suitable double covers of $\mathbb{P}^2$ branched above six lines. We recall such a $K3$ surface $\mathcal{E}_2$, labelled as $\mathcal{A}(2)$ in [SB] by Stienstra and Beukers. It is given by replacing the variable τ in the equation $X(Y-Z)(Z-X)-\tau(X-Y)YZ=0$ by ${t_2}^2$, yielding a two-fold cover of $\mathbb{P}^2$ branched over the six lines $X=0, Y=0, Z=0, X-Y=0, Y-Z=0, Z-X=0$ positioned as follows:



By letting $X=-{t_2}^2VW, Y=-{t_2}^2UW+U^2, Z=-UV$ and further by $x=U/W, y=V/W$, Stienstra and Beukers [SB] arrived at the following nonhomogeneous model for the $K3$ surface $\mathcal{E}_2$ in the sense of Shioda [Shi2]

$$\mathcal{E}_2 \ : \ y^2+(1-{t_2}^2)xy-{t_2}^2y=x^3-{t_2}^2x^2,$$

where t_2 is a parameter. We extend this setting to the family of elliptic surfaces $\mathcal{E}_n$, for $n\geq 2$, in the sense of Shioda defined by

$$\mathcal{E}_n \ : \ y^2+(1-{t_n}^n)xy-{t_n}^ny=x^3-{t_n}^nx^2 \tag{5.1}$$

with $t_n := \sqrt[n]{\tau}$ as a parameter. It is an n-fold cover of $\mathbb{P}^2$ branched above the same configuration of six lines. The Hodge diamond of $\mathcal{E}_n$ is of the form

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ (n-1) & & 10n & & (n-1) \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The action of the Galois group $G_{\mathbb{Q}}$ on the $(12n-2)$-dimensional space $H^2_{et}(\mathcal{E}_n\otimes_{\mathbb{Q}}\overline{\mathbb{Q}},\mathbb{Q}_\ell)$ is an ℓ-adic representation $\tilde{\rho}_{n,\ell}$. As ℓ varies, they form a compatible system such that, at a prime p where $\mathcal{E}_n$ has good reduction, $\tilde{\rho}_{n,\ell}$ for $\ell\neq p$ is unramified at p and $\tilde{\rho}_{n,\ell}(\mathrm{Frob}_p)$ has characteristic polynomial $P_2(\mathcal{E}_n,p,T)\in\mathbb{Z}[T]$ independent of ℓ and of degree $12n-2$. It occurs in the denominator of the Zeta function of $\mathcal{E}_n$ mod p:

$$Z(\mathcal{E}_n/\mathbb{F}_p,T)=\frac{1}{(1-T)(1-p^2T)P_2(\mathcal{E}_n,p,T)}.$$

Moreover, the algebraic cycles on $\mathcal{E}_n$ generate the Néron-Severi group of rank $10n$; its orthogonal complement in the second singular cohomology group of $\mathcal{E}_n$ is a group of rank $2n-2$ generated by transcendental cycles. Each group gives rise to a subrepresentation of $\tilde{\rho}_{n,\ell}$, denoted $\tilde{\rho}_{n,\ell,a}$ (algebraic part) and $\tilde{\rho}_{n,\ell,t}$ (transcendental part), respectively, so that $\tilde{\rho}_{n,\ell}=\tilde{\rho}_{n,\ell,a}\oplus\tilde{\rho}_{n,\ell,t}$. Therefore $P_2(\mathcal{E}_n,p,T)$ is a product of $10n$ linear factors in $\mathbb{Z}[T]$ from counting points on algebraic cycles and a degree $2n-2$

polynomial $Q(\mathcal{E}_n; p; T) \in \mathbb{Z}[T]$ from counting points on transcendental cycles. The system of $(2n-2)-$dimensional ℓ-adic representations $\{\tilde{\rho}_{n,\ell,t}\}$ of $G_{\mathbb{Q}}$ is compatible. The factor at a good prime p of the associated L-function is $1/Q(\mathcal{E}_n; p, p^{-s})$.

5.2. **Fibration of $\mathcal{E}_n$ over a modular curve X_n.** It was shown in [SB] that the elliptic surface $\mathcal{E}$ defined by

$$y^2 + (1-\tau)xy - \tau y = x^3 - \tau x^2$$

with parameter τ is fibered over the genus 0 modular curve $X_{\Gamma^1(5)}$ of the congruence subgroup

$$\Gamma^1(5) \quad = \quad \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \mod 5 \right\}$$

of $\mathrm{SL}_2(\mathbb{Z})$. Thus $\mathcal{E}_n$ is fibered over a genus 0 n-fold cover X_n of $X_{\Gamma^1(5)}$ under $\tau = t_n{}^n$. We give more details about X_n. The curve $X_{\Gamma^1(5)}$ is defined over $\mathbb{Q}$, containing no elliptic points and four cusps, at ∞, 0, -2 and $-5/2$, among them the cusps ∞ and -2 are defined over $\mathbb{Q}$. Let E_1 and E_2 be two Eisenstein series of weight 3 with $\mathbb{Q}$-rational Fourier coefficients and having simple zeros at all cusps except ∞ and -2, respectively. Then $\tau = \frac{E_1}{E_2}$ is a Hauptmodul for $\Gamma^1(5)$ with a simple zero at the cusp -2 and a simple pole at the cusp ∞. With $t_n = \sqrt[n]{\tau}$, the curve X_n is unramified over $X_{\Gamma^1(5)}$ except totally ramified above the cusps ∞ and -2 (with τ-coordinates ∞ and 0, resp.). This describes the index-n normal subgroup Γ_n of $\Gamma^1(5)$ such that X_n is the modular curve of Γ_n. See [ALL] for an expression of Γ_n in terms of generators and relations. Note that $\mathcal{E}_n$ is a universal elliptic curve over X_n.

It is known that Γ_n is a noncongruence subgroup of $SL_2(\mathbb{Z})$ if $n \neq 5$, and Γ_5 is isomorphic to the principal congruence subgroup $\Gamma(5)$. The space of weight 3 cusp forms for Γ_n is $(n-1)$-dimensional, corresponding to holomorphic 2-differentials on $\mathcal{E}_n$; it has an explicit basis given by $(E_1^j E_2^{n-j})^{1/n}$ for $1 \le j \le n-1$ (cf. [ALL,LLY]).

5.3. **Scholl representations attached to $S_3(\Gamma_n)$.** As reviewed in Section 3, to $S_3(\Gamma_n)$ Scholl has attached a compatible system of $2(n-1)$-dimensional ℓ-adic representations $\rho_{n,\ell}$ of $G_{\mathbb{Q}}$ acting on the parabolic cohomology group

$$W_{n,\ell} = H^1(X_n \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \iota_* \mathcal{F}_\ell)$$

of X_n, similar to Deligne's construction of ℓ-adic Galois representations attached to congruence forms (cf. [Sch1]). He also proved in [Sch2] the existence of a Kuga-Sato variety Y_n over $\mathbb{Q}$ of dimension 2 such that $W_{n,\ell}$ can be embedded into the $G_{\mathbb{Q}}$-module $H^2_{et}(Y_n \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$. In our case Y_n is nothing but $\mathcal{E}_n$ and the subrepresentation of $\tilde{\rho}_{n,\ell}$ isomorphic to $\rho_{n,\ell}$ is precisely $\tilde{\rho}_{n,\ell,t}$. Hence $L(\{\rho_{n,\ell}\}, s)$ agrees with $L(\{\tilde{\rho}_{n,\ell,t}\}, s)$.

We list some key properties of the Scholl representations $\rho_{n,\ell}$:

(1) $\rho_{n,\ell}$ is unramified outside $n\ell$.
(2) For ℓ large, $\rho_{n,\ell}|_{G_{\mathbb{Q}_\ell}}$ is crystalline with Hodge-Tate weights 0 and -2, each with multiplicity $n-1$.
(3) $\rho_{n,\ell}$ at the complex conjugation has eigenvalues ± 1, each with multiplicity $n-1$.

(4) The space $W_{n,\ell}$ of $\rho_{n,\ell}$ admits the action by $\zeta = \left(\begin{smallmatrix} 1 & 5 \\ 0 & 1 \end{smallmatrix}\right)$, induced from the action

$$\zeta(t_n) = \zeta_n^{-1} t_n \tag{5.2}$$

on rational functions on X_n. Here $\zeta_n = e^{2\pi\sqrt{-1}/n}$.

The purpose of Section 5 is to prove the (potential) automorphy of the Scholl representation $\rho_{n,\ell}$. Since Γ_5 is a congruence subgroup, $\rho_{5,\ell}$ is naturally automorphic. Our concern is for the case $n \neq 5$. To proceed, we make the following observation. Given $n \geq 2$, it follows from the property (4) above that the action of ζ^* on $W_{n,\ell}$ induced from ζ has order n, so $W_{n,\ell}$ decomposes into the direct sum of eigenspaces of ζ^* with eigenvalues ζ_n^m for $m = 1, ..., n-1$. For a proper divisor d of n with $d > 1$, since $t_d = t_n^{n/d}$, the subspace of $W_{n,\ell}$ on which $(\zeta^*)^d$ acts trivially can be identified with $W_{d,\ell}$ so that $\rho_{d,\ell}$ may be regarded as a subrepresentation of $\rho_{n,\ell}$. The space $W_{d,\ell}$ is the sum of eigenspaces with eigenvalues $\zeta_d^r = \zeta_n^{rn/d}$ for $r = 1, ..., d-1$. Consequently the sum of eigenspaces in $W_{n,\ell}$ with eigenvalues ζ_n^m where $(m, n) = 1$ complements the space $\sum_{d|n, 1<d<n} W_{d,\ell}$. This $G_{\mathbb{Q}}$-invariant subspace is the "new" part of $W_{n,\ell}$, denoted by $\rho_{n,\ell}^{new}$. By inclusion-exclusion, we find that $\rho_{n,\ell}^{new}$ has dimension $2\phi(n)$ (where ϕ is Euler's totient-function) and $\rho_{n,\ell}$ can be decomposed as the sum

$$\rho_{n,\ell} = \bigoplus_{d|n,\ d>1} \rho_{d,\ell}^{new}.$$

Clearly, $\rho_{n,\ell} = \rho_{n,\ell}^{new}$ when n is a prime. It suffices to show that each $\rho_{n,\ell}^{new}$ is (potentially) automorphic. For this, we shall prove the following.

Theorem 5.1. *Let $n \geq 2$ be an integer. There are 2-dimensional ℓ-adic representations σ_ℓ of $G_{\mathbb{Q}(\zeta_n)}$ whose semi-simplifications form a compatible system such that $\rho_{n,\ell}^{new} \cong Ind_{G_{\mathbb{Q}(\zeta_n)}}^{G_{\mathbb{Q}}} \sigma_\ell$ for all primes ℓ. For each ℓ the representation $\rho_{n,\ell}^{new}$ is potentially automorphic. Further, it is automorphic if either $n \leq 6$ or σ_ℓ is potentially reducible.*

We recall the known result in the literature that $\rho_{n,\ell}^{new}$ is automorphic for $n \leq 6$ and $n \neq 5$. First, 2-dimensional Scholl representations of $G_{\mathbb{Q}}$ are automorphic by Theorem 2.2. In particular, $\rho_{2,\ell}$ is modular. In fact Stienstra and Beukers have already shown in [SB] that $\tilde{\rho}_{2,\ell,t}$ ($\cong \rho_{2,\ell}$) is isomorphic to the ℓ-adic Deligne representation attached to the congruence cusp form $\eta(4\tau)^6$. Using the method of Faltings-Serre, Li, Long, and Yang proved in [LLY] the automorphy of the 4-dimensional $\rho_{3,\ell}$, which corresponds to an automorphic representation of $\mathrm{GL}_2 \oplus \mathrm{GL}_2$ over $\mathbb{Q}$. When $n = 4$, it was observed in [ALL] that the space of $\rho_{4,\ell}^{new}$ admits quaternion multiplication. Using these symmetries they showed that the representation $\rho_{4,\ell}^{new}$ is automorphic, corresponding to an automorphic representation of $\mathrm{GL}_2 \times \mathrm{GL}_2$ over $\mathbb{Q}$, and hence also an automorphic representation of GL_4 over $\mathbb{Q}$ by a result of Ramakrishnan [Ram]. Thus $\rho_{4,\ell}$ is automorphic. The same holds for $\rho_{6,\ell}^{new}$ by a similar argument carried out by Long [Lon]. Thus $\rho_{6,\ell}$ is automorphic. The paper [AL^3] by Atkin, Li, Liu, and Long gives a conceptual explanation of the automorphy of 4-dimensional Galois representations with QM, including the cases $n = 3, 4, 6$.

5.4. **The structure of $\rho_{n,\ell}^{new}$.** Since the action of ζ is defined over the cyclotomic field $\mathbb{Q}(\zeta_n)$, each eigenspace of ζ is $G_{\mathbb{Q}(\zeta_n)}$-invariant. Denote the subrepresentation of $\rho_{n,\ell}|_{G_{\mathbb{Q}(\zeta_n)}}$ on the eigenspace with eigenvalue ζ_n^i by $\sigma_{n,\ell,i}$ so that

$$\rho_{n,\ell}|_{G_{\mathbb{Q}(\zeta_n)}} = \bigoplus_{1\le i\le n-1} \sigma_{n,\ell,i}$$

and

$$\rho_{n,\ell}^{new}|_{G_{\mathbb{Q}(\zeta_n)}} = \bigoplus_{1\le i\le n-1,\ (i,n)=1} \sigma_{n,\ell,i}.$$

As computed in [SB], the model (5.1) for $\mathcal{E}_n$ came from the homogenous model

$$X(Y-Z)(Z-X) - t_n^n(X-Y)YZ = 0 \tag{5.3}$$

for the surface by setting $X = -t_n^n y, Y = -t_n^n x - x^2, Z = -xy$. If, instead, we let $x = \frac{Z}{X}, y = \frac{Y}{Z}$, and $s = t_n \cdot \frac{X^2}{Y(X-Y)}$, then we get the following model:

$$\mathcal{E}_n \ : \ s^n = (xy)^{n-1}(1-y)(1-x)(1-xy)^{n-1} =: f_n(x,y) \tag{5.4}$$

which is more amenable to our computation. Observe that the action of ζ on t_n in (5.2) translates to $\zeta(s) = \zeta_n^{-1}s$ for the model (5.4).

Given $g \in G_{\mathbb{Q}}$, its action on $\mathbb{Q}(\zeta_n)$ is determined by the image $g(\zeta_n) = \zeta_n^{\varepsilon(g)}$, where the exponent $\varepsilon(g)$ lies in $(\mathbb{Z}/n\mathbb{Z})^\times$. For $(x,y,s) \in \mathcal{E}_n(\overline{\mathbb{Q}})$ defined by the model (5.4), we have

$$g\circ\zeta(x,y,s) = g(x,y,\zeta_n^{-1}s) = (g(x),g(y),g(\zeta_n)^{-1}g(s)) = \zeta^{\varepsilon(g)}\circ g(x,y,s),$$

that is, $g\circ\zeta = \zeta^{\varepsilon(g)}\circ g$. Since the space of $\rho_{n,\ell}$ is contained in $H^2_{et}(\mathcal{E}_n \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$, this relation between g and ζ implies that the induced action of g on $\rho_{n,\ell}$ sends the ζ_n^i-eigenspace of ζ to the $g(\zeta_n)^i$-eigenspace of ζ. Therefore the conjugate of $\sigma_{n,\ell,i}$ by g is isomorphic to $\sigma_{n,\ell,\varepsilon(g)i}$. Consequently, for a fixed n, the $\sigma_{n,\ell,i}$'s with $(i,n)=1$ are conjugates of $\sigma_{n,\ell,1}$ by $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, hence they are all 2-dimensional and $\rho_{n,\ell}^{new} = \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{\mathbb{Q}}}\sigma_{n,\ell,1}$. Further, for the complex conjugation c in $G_{\mathbb{Q}}$, we have $\varepsilon(c) = -1$ so that, for $1\le i\le n-1$,

$$\sigma_{n,\ell,i}^c = \sigma_{n,\ell,n-i}.$$

We record the above discussion in the following.

Lemma 5.2. (1) *For $g\in G_{\mathbb{Q}}$ and $1\le i\le n-1$, we have $\sigma_{n,\ell,i}^g \cong \sigma_{n,\ell,\varepsilon(g)i}$, where $\varepsilon(g)$ is such that $g(\zeta_n) = \zeta_n^{\varepsilon(g)}$. Therefore all $\sigma_{n,\ell,i}$ are 2-dimensional, and*

$$\rho_{n,\ell}^{new}|_{G_{\mathbb{Q}(\zeta_n)}} = \bigoplus_{g\in\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})} \sigma_{n,\ell,i}^g \quad \text{and} \quad \rho_{n,\ell}^{new} \cong \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{\mathbb{Q}}}\sigma_{n,\ell,i}$$

for any i coprime to n.

(2) *For the complex conjugation $c\in G_{\mathbb{Q}}$, we have $\sigma_{n,\ell,i}^c \cong \sigma_{n,\ell,n-i}$.*

5.5. **Computing the trace of $\sigma_{n,\ell,i}$.** The aim of this subsection is to express the trace of $\sigma_{n,\ell,i}$ at Frobenius elements in terms of character sums by using the isomorphism $\rho_{n,\ell}\cong\tilde{\rho}_{n,\ell,t}$. As a result, the semi-simplification of $\sigma_{n,\ell,i}$, as ℓ varies, forms a compatible system.

Let K be a finite extension of the cyclotomic field $\mathbb{Q}(\zeta_n)$. At a place $\mathfrak{p}$ of K not dividing n with residue field $k_{\mathfrak{p}}$ of cardinality $N\mathfrak{p}$, we have $n|N\mathfrak{p}-1$. The nth power residue symbol at $\mathfrak{p}$, denoted by $\left(\frac{\cdot}{\mathfrak{p}}\right)_n$, is the $\langle\zeta_n\rangle \cup \{0\}$-valued function defined by

$$\left(\frac{a}{\mathfrak{p}}\right)_n \equiv a^{(N\mathfrak{p}-1)/n} \pmod{\mathfrak{p}} \quad \text{for all } a \in \mathbb{Z}_K,$$

where $\mathbb{Z}_K$ denotes the ring of integers of K. It induces a character $\xi_{\mathfrak{p},n}$ of order n on $k_{\mathfrak{p}}^{\times}$ extended to $k_{\mathfrak{p}}$ by $\xi_{\mathfrak{p},n}(0) = 0$. Also for fixed nonzero $a \in \mathbb{Z}_K$, as $\mathfrak{p}$ varies among the finite places of K not dividing na, the power residue symbol defines a representation of the Galois group $\mathrm{Gal}(K(\sqrt[n]{a})/K)$ such that

$$\left(\frac{a}{\mathfrak{p}}\right)_n = \frac{\mathrm{Fr}_{\mathfrak{p}}(\sqrt[n]{a})}{\sqrt[n]{a}}, \tag{5.5}$$

where $\mathrm{Fr}_{\mathfrak{p}}$ is the the arithmetic Frobenius at $\mathfrak{p}$. This fact, originally due to Hilbert, is part of the class field theory. See, for example, [FLRST, sections 5 and 6] for more detail.

We analyze the rational points on $\mathcal{E}_n$ using the model (5.4) given by

$$\mathcal{E}_n \ : \ s^n = (xy)^{n-1}(1-y)(1-x)(1-xy)^{n-1} =: f_n(x,y).$$

The solutions to the above equation with $s = 0$ lie on algebraic cycles. At a place $\mathfrak{p}$ of K not dividing n, $\mathcal{E}_n$ has good reduction mod $\mathfrak{p}$ and the number of solutions to (5.4) mod $\mathfrak{p}$ with $s \neq 0$ can be expressed in terms of the character sum

$$\sum_{1\le i\le n} \sum_{x,y\in k_{\mathfrak{p}}} \xi_{\mathfrak{p},n}^i(f_n(x,y)). \tag{5.6}$$

Noticing that $\xi_{\mathfrak{p},n}$ has order n, we can rewrite the inner sum as

$$\sum_{\substack{x,y\in k_{\mathfrak{p}}^{\times} \\ x\neq 1, y\neq 1, xy\neq 1}} \xi_{\mathfrak{p},n}^i(g(x,y)),$$

where $g(x,y) = \frac{(1-x)(1-y)}{xy(1-xy)}$ is independent of n. As $\xi_{\mathfrak{p},n}^i$ has order $n/(n,i)$, the sum over i with $(n,i) = d < n$ first occurs in the sum for $\mathcal{E}_{n/d}$. Further, the inner sum with $i = n$ contributes to the factors of the zeta function of $\mathcal{E}_n$ mod $\mathfrak{p}$ other than $Q(\mathcal{E}_n; \mathfrak{p}, T)$, and the sum in (5.6) over $1 \le i < n$ counts the number of $k_{\mathfrak{p}}$-rational points on the transcendental cycles mod $\mathfrak{p}$, hence it is equal to the coefficient of the second highest order term in $Q(\mathcal{E}_n; \mathfrak{p}, T)$. This shows that, inductively, for $\mathfrak{p}$ not dividing ℓn, we have

$$\mathrm{Tr}\,\rho_{n,\ell}|_{G_K}(\mathrm{Frob}_{\mathfrak{p}}) = \mathrm{Tr}\,\tilde{\rho}_{n,\ell,t}|_{G_K}(\mathrm{Frob}_{\mathfrak{p}}) = \sum_{1\le i<n} \sum_{x,y\in k_{\mathfrak{p}}} \xi_{\mathfrak{p},n}^i(f_n(x,y)).$$

Moreover, the sum over $1 \le i < n$ with $(i,n) = 1$ counts those points on $\mathcal{E}_n$ which are not contained in $\mathcal{E}_d$ with d dividing n properly, so it is the trace of the "new" part of $\tilde{\rho}_{n,\ell,t}$ evaluated at $\mathrm{Frob}_{\mathfrak{p}}$, i.e.,

$$\mathrm{Tr}\,\rho_{n,\ell}^{new}|_{G_K}(\mathrm{Frob}_{\mathfrak{p}}) = \mathrm{Tr}\,\tilde{\rho}_{n,\ell,t}^{new}|_{G_K}(\mathrm{Frob}_{\mathfrak{p}}) = \sum_{\substack{1\le i<n \\ (n,i)=1}} \sum_{x,y\in k_{\mathfrak{p}}} \xi_{\mathfrak{p},n}^i(f_n(x,y)). \tag{5.7}$$

Both integers are independent of the auxiliary prime ℓ.

In [DFLST] the authors considered certain families of generalized Legendre curves whose associated Galois representations have a similar decomposition as a sum of new parts. It was shown there that each new part restricted to the Galois group of a suitable cyclotomic field further decomposes into a direct sum of degree-2 representations whose trace at the Frobenius elements at unramified places are in fact character sums occurring in counting rational points of the underlying curve over the residue fields. This result is further extended in [FLRST] to more general curves. In the theorem below we prove that an analogous result holds for restrictions of our 2-dimensional representations $\sigma_{n,\ell,i}$ to finite index subgroups of $G_{\mathbb{Q}(\zeta_n)}$. The argument below follows §6.3 of [FLRST].

Theorem 5.3. *Fix $n \geq 2$ and ℓ. Let K be a finite extension of $\mathbb{Q}(\zeta_n)$. Then $\sigma_{n,\ell,i}|_{G_K}$ for $1 \leq i \leq n-1$ are unramified at each place $\mathfrak{p}$ of K not dividing ℓn and satisfy*

$$\mathrm{Tr}\ \sigma_{n,\ell,i}|_{G_K}(\mathrm{Frob}_{\mathfrak{p}}) = \sum_{x,y \in k_{\mathfrak{p}}} \xi^i_{\mathfrak{p},n}(f_n(x,y)).$$

Proof. First we remark that it suffices to prove the theorem for all pairs $\{n,i\}$ with $(i,n)=1$. This is because for the case $(i,n)=d>1$, we have $\sigma_{n,\ell,i} \cong \sigma_{n/d,\ell,i/d}$ and $\xi^i_{\mathfrak{p},n}(f_n(x,y)) = \xi^{i/d}_{\mathfrak{p},n/d}(f_{n/d}(x,y))$ as observed before so that the statement in this case is the same as that with the pair $\{n,i\}$ replaced by $\{n/d, i/d\}$. Therefore we assume $(i,n)=1$ in the argument below.

Let $\mathfrak{p}$ be a place of K not dividing $n\ell$ with residue field $k_{\mathfrak{p}}$. Then $\rho_{n,\ell}|_{G_K}$ and hence $\sigma_{n,\ell,i}|_{G_K}$ are unramified at $\mathfrak{p}$ for all i. Choose an element $c \in \mathbb{Z}_K$ such that $\xi_{\mathfrak{p},n}(c) = \zeta_n$ is a primitive nth root of unity. Then $F = K(\sqrt[n]{c})$ is an abelian extension of K of degree at most n since K contains ζ_n. It follows from (5.5) that the Frobenius element $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(F/K)$, being the inverse of the arithmetic Frobenius $\mathrm{Fr}_{\mathfrak{p}}$ at $\mathfrak{p}$, maps $\sqrt[n]{c}$ to $\xi_{\mathfrak{p},n}(c)^{-1}\sqrt[n]{c} = \zeta_n^{-1}\sqrt[n]{c}$, hence it has order n. Therefore F is a cyclic degree n extension of K and $\mathrm{Fr}_{\mathfrak{p}}$ generates $\mathrm{Gal}(F/K)$. The dual of $\mathrm{Gal}(F/K)$ is generated by the character ξ_c satisfying $\xi_c(\mathrm{Fr}_{\mathfrak{p}}) = \left(\frac{c}{\mathfrak{p}}\right)_n = \zeta_n$.

For each integer $r \geq 0$, consider the twist $\mathcal{T}_{n,c,r}$ of $\mathcal{E}_n$ over K by c^r defined by

$$\mathcal{T}_{n,c,r}\ :\ s^n = c^r f_n(x,y).$$

Note that $\mathcal{T}_{n,c,0} = \mathcal{E}_n$. Denote by $\tilde{\rho}_{n,\ell,c,r,t}$ the ℓ-adic G_K-action on the transcendental lattice of $\mathcal{T}_{n,c,r}$ and by $\tilde{\rho}^{new}_{n,\ell,c,r,t}$ its new part. The same argument as before shows that

$$\mathrm{Tr}\,\tilde{\rho}^{new}_{n,\ell,c,r,t}(\mathrm{Frob}_{\mathfrak{p}}) = \sum_{1 \leq i < n,\ (n,i)=1}\ \sum_{x,y \in k_{\mathfrak{p}}} \xi^i_{\mathfrak{p},n}(c^r f_n(x,y)) \tag{5.8}$$

$$= \sum_{1 \leq i < n,\ (n,i)=1} \xi^i_{\mathfrak{p},n}(c^r) \sum_{x,y \in k_{\mathfrak{p}}} \xi^i_{\mathfrak{p},n}(f_n(x,y)). \tag{5.9}$$

The map $T\ :\ (x,y,s) \mapsto (x,y,\sqrt[n]{c}s)$ yields an isomorphism over F from $\mathcal{T}_{n,c,r}$ to $\mathcal{T}_{n,c,r+1}$ for $r \geq 0$. Therefore $\tilde{\rho}_{n,\ell,c,r,t}|_{G_F} \cong \rho_{n,\ell}|_{G_F}$ and the same holds for their new part:

$$\tilde{\rho}^{new}_{n,\ell,c,r,t}|_{G_F} \cong \rho^{new}_{n,\ell}|_{G_F}. \tag{5.10}$$

On the representation spaces, T induces a map $T^* : \tilde{\rho}^{new}_{n,\ell,c,r+1,t} \to \tilde{\rho}^{new}_{n,\ell,c,r,t}$. Further, the map $\zeta\ :\ (x,y,s) \mapsto (x,y,\zeta_n^{-1}s)$ is an automorphism on $\mathcal{T}_{n,c,r}$ defined over $K \supset \mathbb{Q}(\zeta_n)$, hence it induces an operator ζ^* of order n on the representation space of $\tilde{\rho}_{n,\ell,c,r,t}$. This operator decomposes $\tilde{\rho}^{new}_{n,\ell,c,r,t}$ into a direct sum of $\phi(n)$ subrepresentations $\tau_{n,\ell,c,r,i}$, where $1 \leq i \leq n-1$ and $(n,i)=1$, acting on the eigenspace of ζ^* with eigenvalue ζ_n^i. Since ζ and T commute over F, T^* yields an isomorphism from the ζ_n^i-eigenspace of ζ^* on $\tilde{\rho}_{n,\ell,c,r+1,t}^{new}$ to that on $\tilde{\rho}_{n,\ell,c,r,t}^{new}$. Combined with (5.10), we obtain

$$\sigma_{n,\ell,i}|_{G_F} \cong \tau_{n,\ell,c,r,i}|_{G_F}$$

for all $1 \leq i \leq n-1$ coprime to n.

On the other hand, for $(x,y,s) \in \mathcal{T}_{n,c,r}(\overline{\mathbb{Q}})$, by (5.5) we have

$$\begin{aligned} \mathrm{Frob}_{\mathfrak{p},r+1} \circ T(x,y,s) &= \mathrm{Frob}_{\mathfrak{p},r+1}(x,y,\sqrt[n]{c}s) \\ &= (\mathrm{Frob}_{\mathfrak{p},r}(x), \mathrm{Frob}_{\mathfrak{p},r}(y), \zeta_n^{-1}\sqrt[n]{c}\ \mathrm{Frob}_{\mathfrak{p},r}(s)) \\ &= T \circ \zeta \circ \mathrm{Frob}_{\mathfrak{p},r}(x,y,s). \end{aligned}$$

Here, for the sake of clarity, we put subscripts on $\mathrm{Frob}_{\mathfrak{p}}$ to keep track of the spaces on which it acts. Therefore, on the space of $\tilde{\rho}_{n,\ell,c,r,t}^{new}$ we have $T^* \circ \mathrm{Frob}_{\mathfrak{p},r+1} \circ (T^*)^{-1} = \mathrm{Frob}_{\mathfrak{p},r} \circ \zeta^*$. Here, by abuse of notation, we use $\mathrm{Frob}_{\mathfrak{p},r}$ to denote the action on $\tilde{\rho}_{n,\ell,c,r,t}^{new}$ induced by the geometric Frobenius at $\mathfrak{p}$. By construction, ζ^* acts on the representation spaces of $\tau_{n,\ell,c,r,i}$ and $\tau_{n,\ell,c,r,i}|_{G_F}$ via multiplication by $\zeta_n^i = \xi_{\mathfrak{p},n}(c)^i$; this shows that

$$\mathrm{Tr}\ \tau_{n,\ell,c,r+1,i}(\mathrm{Frob}_{\mathfrak{p},r+1}) = \xi_{\mathfrak{p},n}(c)^i\ \mathrm{Tr}\ \tau_{n,\ell,c,r,i}(\mathrm{Frob}_{\mathfrak{p},r})$$

and recursively this gives

$$\mathrm{Tr}\ \tau_{n,\ell,c,r,i}(\mathrm{Frob}_{\mathfrak{p},r}) = \xi_{\mathfrak{p},n}(c^r)^i\ \mathrm{Tr}\ \sigma_{n,\ell,i}|_{G_K}(\mathrm{Frob}_{\mathfrak{p}}).$$

Consequently we obtain, for $r \geq 0$,

$$\mathrm{Tr}\,\tilde{\rho}_{n,\ell,c,r,t}^{new}(\mathrm{Frob}_{\mathfrak{p},r}) = \sum_{1 \leq i < n,\ (n,i)=1} \xi_{\mathfrak{p},n}(c^r)^i\ \mathrm{Tr}\ \sigma_{n,\ell,i}|_{G_K}(\mathrm{Frob}_{\mathfrak{p}}). \tag{5.11}$$

Compare this with (5.7) for $0 \leq r \leq \phi(n)-1$. We regard both as a system of $\phi(n)$ linear equations whose coefficient matrix is a nonsingular Vandermonde matrix $\big(\xi_{\mathfrak{p},n}(c^r)^i\big)_{\substack{0 \leq r \leq \phi(n)-1 \\ 1 \leq i < n, (n,i)=1}}$ (because $\xi_{\mathfrak{p},n}(c) = \zeta_n$ has order n). Hence the system has a unique solution, from which it follows that

$$\mathrm{Tr}\ \sigma_{n,\ell,i}|_{G_K}(\mathrm{Frob}_{\mathfrak{p}}) = \sum_{x,y \in k_{\mathfrak{p}}} \xi_{\mathfrak{p},n}^i(f_n(x,y)),$$

as desired. □

Applying Theorem 5.3 to $K = \mathbb{Q}(\zeta_n)$, we get that, for each place $\mathfrak{p}$ of $\mathbb{Q}(\zeta_n)$ not dividing $n\ell$, $\mathrm{Tr}\ \sigma_{n,\ell,i}(\mathrm{Frob}_{\mathfrak{p}}) \in \mathbb{Q}(\zeta_n)$ is independent of ℓ. Further, for such $\mathfrak{p}$ there is a quadratic extension $K(\mathfrak{p})$ of $\mathbb{Q}(\zeta_n)$ which has only one place $\wp$ above $\mathfrak{p}$. Applying Theorem 5.3 to $K = K(\mathfrak{p})$, we conclude that $\mathrm{Tr}\ \sigma_{n,\ell,i}|_{G_{K(\mathfrak{p})}}(\mathrm{Frob}_{\wp}) \in \mathbb{Q}(\zeta_n)$ is also independent of ℓ. The same holds for

$$\det \sigma_{n,\ell,i}(\mathrm{Frob}_{\mathfrak{p}}) = \frac{1}{2}\big((\mathrm{Tr}\ \sigma_{n,\ell,i}(\mathrm{Frob}_{\mathfrak{p}}))^2 - \mathrm{Tr}\ \sigma_{n,\ell,i}|_{G_{K(\mathfrak{p})}}(\mathrm{Frob}_{\wp})\big).$$

Combined, this shows that for all finite places $\mathfrak{p}$ of $\mathbb{Q}(\zeta_n)$ not dividing n, the characteristic polynomial of $\sigma_{n,\ell,i}(\mathrm{Frob}_{\mathfrak{p}})$ is independent of ℓ not divisible by $\mathfrak{p}$. In view of the definition of a compatible system of Galois representations in §2.4, this proves the following.

Corollary 5.4. *For each $1 \leq i \leq n-1$, the semi-simplifications of $\sigma_{n,\ell,i}$ form a compatible system of ℓ-adic representations of $G_{\mathbb{Q}(\zeta_n)}$. Further, all $\mathrm{Tr}\ \sigma_{n,\ell,i}$ are $\mathbb{Q}(\zeta_n)$-valued and $\mathrm{Tr}\ \sigma_{n,\ell,i}$ is obtained from $\mathrm{Tr}\ \sigma_{n,\ell,1}$ with ζ_n replaced by ζ_n^i.*

Another consequence of Theorem 5.3 is the following character sum estimate, resulting from the fact that the 2-dimensional representation $\sigma_{n,\ell,i}$ is contained in the second étale cohomology of $\mathcal{E}_n$ and the Riemann Hypothesis for the reduction of $\mathcal{E}_n$ at a good place holds.

Corollary 5.5. *Let K be a finite extension of $\mathbb{Q}(\zeta_n)$. Let $\mathfrak{p}$ be a finite place of K not dividing n whose residue field $k_\mathfrak{p}$ has cardinality $N\mathfrak{p}$. For $1 \le i \le n-1$ coprime to n we have*

$$|\sum_{x,y\in k_\mathfrak{p}} \xi^i_{\mathfrak{p},n}(f_n(x,y))| \le 2N\mathfrak{p}.$$

At a place $\mathfrak{p}$ of $\mathbb{Q}(\zeta_n)$ not dividing n with residue field $k_\mathfrak{p}$, we can express

$$\begin{aligned}\operatorname{Tr}\sigma_{n,\ell,i}(\mathrm{Frob}_\mathfrak{p}) &= \sum_{x,y\in k_\mathfrak{p}} \xi^i_{\mathfrak{p},n}(f_n(x,y))\\ &= \sum_{x,y\in k_\mathfrak{p}} \xi^{-i}_{\mathfrak{p},n}(x)\xi^{i}_{\mathfrak{p},n}(1-x)\xi^{-i}_{\mathfrak{p},n}(y)\xi^{i}_{\mathfrak{p},n}(1-y)\xi^{-i}_{\mathfrak{p},n}(1-xy),\end{aligned}$$

which, by Corollary 3.14 (i) of Greene [Gre], is equal to $|k_\mathfrak{p}|^2$ times the hypergeometric function ${}_3F_2\left(\begin{smallmatrix}\xi^i_{\mathfrak{p},n}, & \xi^{-i}_{\mathfrak{p},n}, & \xi^{-i}_{\mathfrak{p},n}\\ & 1, & 1\end{smallmatrix}|1\right)$ over the finite field $k_\mathfrak{p}$. Using the identities (4.26) and (4.23) of [Gre], we arrive at the relation

$$\sum_{x,y\in k_\mathfrak{p}} \xi^i_{\mathfrak{p},n}(f_n(x,y)) = \left(\frac{-1}{\mathfrak{p}}\right)^i_n \sum_{x,y\in k_\mathfrak{p}} \xi^{n-i}_{\mathfrak{p},n}(f_n(x,y)) \tag{5.12}$$

for all $1 \le i < n$ and finite places $\mathfrak{p}$ of $\mathbb{Q}(\zeta_n)$ not dividing n. By (5.5), the map $\mathrm{Frob}_\mathfrak{p} \mapsto \left(\frac{-1}{\mathfrak{p}}\right)_n$ is a character $\xi_{n,-1}$ of $G_{\mathbb{Q}(\zeta_n)}$. Combined with Theorem 5.3 this gives

$$(\sigma_{n,\ell,i})^{ss} \cong (\sigma_{n,\ell,n-i})^{ss} \otimes \xi^i_{n,-1}. \tag{5.13}$$

Here σ^{ss} denotes the semi-simplification of the representation σ. The kernel of $\xi_{n,-1}$ is $G_{\mathbb{Q}(\zeta_{2n})}$. When n is odd, $\mathbb{Q}(\zeta_{2n}) = \mathbb{Q}(\zeta_n)$ and hence $\xi_{n,-1}$ is trivial; while for n even, $\mathbb{Q}(\zeta_{2n})$ is a quadratic extension of $\mathbb{Q}(\zeta_n)$ so that $\xi_{n,-1}$ has order 2.

Since a semi-simple representation is determined by its trace, we summarize the above discussion below.

Proposition 5.6. *For $(i,n)=1$, $\sigma^{ss}_{n,\ell,i} \cong \sigma^{ss}_{n,\ell,n-i} \otimes \xi^i_{n,-1}$. Consequently $\sigma^{ss}_{n,\ell,i}$ and $\sigma^{ss}_{n,\ell,n-i}$ are equivalent when restricted to $G_{\mathbb{Q}(\zeta_{2n})}$. Further, for $n \ge 3$ odd, we have $\sigma^{ss}_{n,\ell,i} \cong \sigma^{ss}_{n,\ell,n-i}$.*

Remark 6. Proposition 5.6 is derived using identities on character sums given in [Gre]. Another way to get the relation between $\sigma_{n,\ell,i}$ and $\sigma_{n,\ell,n-i}$ is to use the symmetry on the modular curve X_n arising from the operator $A = \left(\begin{smallmatrix}-2 & -5\\ 1 & 2\end{smallmatrix}\right) \in \Gamma^0(5)$ which normalizes Γ_n. By choosing the Hauptmodul $t = E_1/E_2$ with $E_2 = E_1|_A$ on $X_{\Gamma^1(5)}$, A maps t to $-1/t$ and t_n to ζ_{2n}/t_n. The relation $A\zeta = \zeta^{-1}A$ on X_n gives rise to the isomorphism $\sigma_{n,\ell,i} \cong \sigma_{n,\ell,n-i}$ for n odd and $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}} \cong \sigma_{n,\ell,n-i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$ for n even.

5.6. **A proof of Theorem 5.1.** First we deal with reducible $\sigma_{n,\ell,i}$.

Lemma 5.7. *$\rho_{n,\ell}^{new}$ is automorphic if there is some $1 \le i \le n-1$ coprime to n such that either*

(1) *$\sigma_{n,\ell,i}$ is reducible, or*
(2) *$\sigma_{n,\ell,i}$ is irreducible and $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$ is reducible.*

Proof. By Lemma 5.2(1), $\sigma_{n,\ell,i}$ is 2-dimensional and $\rho_{n,\ell}^{new} = \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{\mathbb{Q}}} \sigma_{n,\ell,i}$. We divide the proof into two cases according to the assumptions.

(1) $\sigma_{n,\ell,i}$ is reducible. Then it contains a 1-dimensional subrepresentation χ_1 and its semi-simplification decomposes as $\chi_1 \oplus \chi_2$, where χ_1 and χ_2 are geometric characters of $G_{\mathbb{Q}(\zeta_n)}$. Then $\alpha := \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{\mathbb{Q}}} \chi_1$ is a $\phi(n)$-dimensional subrepresentation of $\rho_{n,\ell}^{new}$. As χ_1 is automorphic and $\mathbb{Q}(\zeta_n)$ is a finite abelian extension of $\mathbb{Q}$, α is automorphic by automorphic induction. The same holds for the quotient $\beta := \rho_{n,\ell}^{new}/\alpha \cong \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{\mathbb{Q}}} \chi_2$. This proves that $\rho_{n,\ell}^{new}$ is automorphic.

(2) By the assumption on $\sigma_{n,\ell,i}$ and Theorem 2.3, $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}} = \xi_1 \oplus \xi_2$ for two regular characters ξ_1 and ξ_2 of $G_{\mathbb{Q}(\zeta_{2n})}$. Note that in this case $\mathbb{Q}(\zeta_{2n})$ is a quadratic extension of $\mathbb{Q}(\zeta_n)$ and n is even. Let g be an element in $G_{\mathbb{Q}(\zeta_n)} \setminus G_{\mathbb{Q}(\zeta_{2n})}$. Then $\xi_1^g = \xi_1$ or ξ_2.

Case (2.1) $\xi_1^g = \xi_2$. Then $\sigma_{n,\ell,i} = \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_{2n})}}^{G_{\mathbb{Q}(\zeta_n)}} \xi_1$ so that $\rho_{n,\ell}^{new} = \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_{2n})}}^{G_{\mathbb{Q}}} \xi_1$ is automorphic by automorphic induction.

Case (2.2) $\xi_1^g = \xi_1$. Then $\xi_2^g = \xi_2$ so that both ξ_1 and ξ_2 extend to characters of $G_{\mathbb{Q}(\zeta_n)}$. This contradicts the irreducibility of $\sigma_{n,\ell,i}$. □

In view of the lemma above, we shall assume that all $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$ with $(i,n)=1$ are absolutely irreducible for the rest of the proof.

When $n=2$, $\rho_{2,\ell}$ is 2-dimensional and odd, hence is automorphic. Moreover, $\rho_{2,\ell} = \sigma_{2,\ell,1} \cong \xi_{2,-1} \otimes \rho_{2,\ell}$ by Proposition 5.6. This implies that $\rho_{2,\ell}$ has CM by $\mathbb{Q}(\sqrt{-1})$, as observed in [SB].

Now we prove the theorem for $n \ge 3$, in which case $\phi(n)$ is even. With n fixed, denote by $F_n := \mathbb{Q}(\zeta_n)^+$ the totally real subfield of the cyclotomic field $\mathbb{Q}(\zeta_n)$. The restriction of the complex conjugation c to $\mathbb{Q}(\zeta_n)$ generates $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n)^+)$. It follows from Lemma 5.2 and Proposition 5.6 that

$$\tau_{n,\ell,i} := \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{F_n}} \sigma_{n,\ell,i} = \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{F_n}} \sigma_{n,\ell,n-i}$$

is potentially 2-isotypic over $\mathbb{Q}(\zeta_{2n})$ and

$$\rho_{n,\ell}^{new} \cong \mathrm{Ind}_{G_{F_n}}^{G_{\mathbb{Q}}} \tau_{n,\ell,i}.$$

To prove the potential automorphy of $\rho_{n,\ell}^{new}$ for $n \ge 3$, our strategy is to use Proposition 3.7. For this purpose, we need to show, for each $1 \le i \le (n-1)/2, (i,n)=1$, the existence of 2-dimensional ℓ-adic representations $\eta_{n,\ell,i}$ and $\eta_{n,\ell,n-i}$ of G_{F_n} so that their restrictions to $G_{\mathbb{Q}(\zeta_n)}$ are finitely projectively equivalent to $\sigma_{n,\ell,i}$ and $\sigma_{n,\ell,n-i}$, respectively. To do this, we divide the argument into two cases, according as $\tau_{n,\ell,i}$ reducible or irreducible. Observe from Lemma 5.2(1) that, for any $g \in G_{\mathbb{Q}}$, $\tau_{n,\ell,i}^g = \tau_{n,\ell,\varepsilon(g)i}$. So the $\tau_{n,\ell,i}$'s will be simultaneously reducible or irreducible.

(i) $\tau_{n,\ell,i}$ is reducible. This includes all odd $n \ge 3$ because in this case $\sigma_{n,\ell,i} \cong \sigma_{n,\ell,i}^c = \sigma_{n,\ell,n-i}$ by Proposition 5.6. Since $\sigma_{n,\ell,i}$ and $\sigma_{n,\ell,n-i}$ are assumed to be irreducible, this forces the semi-simplificiation of $\tau_{n,\ell,i}$ to be $\eta_{n,\ell,i} \oplus \eta_{n,\ell,n-i}$ so that $\eta_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_n)}} \simeq \sigma_{n,\ell,i}$ and $\eta_{n,\ell,n-i} = \eta_{n,\ell,i} \otimes \chi$ with χ the quadratic character associated to $\mathbb{Q}(\zeta_n)/F_n$. (In fact $\tau_{n,\ell,i} = \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{F_n}} \sigma_{n,\ell,i} = \eta_{n,\ell,i} \oplus \eta_{n,\ell,i} \otimes \chi$ in this case.) Now we can apply Corollary 3.9 to conclude that $\rho_{n,\ell}^{new}$ is potentially automorphic, and in fact automorphic when $F_n = \mathbb{Q}$.

(ii) $\tau_{n,\ell,i}$ is irreducible. Then $n \geq 4$ is even. By Proposition 5.6, each $\tau_{n,\ell,i}$ is 2-isotypic over $\mathbb{Q}(\zeta_{2n})$ since $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$ is assumed to be irreducible. According to Theorem 2.3, there exists a 2-dimensional ℓ-adic representation $\eta_{n,\ell,i}$ of G_{F_n} so that $\eta_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$ is finitely projectively equivalent to $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$. We also have $\eta_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}} \simeq \eta_{n,\ell,i}^c|_{G_{\mathbb{Q}(\zeta_{2n})}}$ finitely projectively equivalent to $\sigma_{n,\ell,i}^c|_{G_{\mathbb{Q}(\zeta_{2n})}} = \sigma_{n,\ell,n-i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$. So we may choose $\eta_{n,\ell,n-i} = \eta_{n,\ell,i}$. Thus the requirement of Corollary 3.9 is satisfied and thus $\rho_{n,\ell}^{new}$ is potentially automorphic.

When $n = 3, 4, 6$, the field $F_n = \mathbb{Q}$, $\rho_{n,\ell}^{new}$ is 4-dimensional and it is 2-isotypic over $\mathbb{Q}(\zeta_{2n})$. We can also conclude the automorphy of $\rho_{n,\ell}^{new}$ from Theorem 4.4. For $n = 3$, the argument in (i) above shows that $\rho_{3,\ell}^{new}$ is the sum of two degree-2 automorphic representations which differ by the quadratic twist χ_{-3} attached to $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$, as shown in [LLY]. When $n = 4$ and 6, Clifford theory (Theorem 2.3) implies that $\rho_{n,\ell}^{new}$ corresponds to an automorphic representation of $\mathrm{GL}_2 \times \mathrm{GL}_2$ over $\mathbb{Q}$, as explained in [AL^3]. When $n = 5$, as remarked before, the group Γ_5 is isomorphic to the congruence subgroup $\Gamma(5)$, hence $\rho_{5,\ell}$ is automorphic.

To complete the proof of Theorem 5.1, it remains to show the following.

Proposition 5.8. *If $\sigma_{n,\ell,i}$ is potentially reducible for some i coprime to n, then $\rho_{n,\ell}^{new}$ is automorphic.*

Proof. In view of Lemma 5.7, we may assume that $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$ is irreducible and it is potentially reducible. From the discussion of cases (i) and (ii) above, there exists a representation $\eta_{n,\ell,i}$ of G_{F_n} so that $\eta_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$ is finitely projectively equivalent to $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$. So $\eta_{n,\ell,i}$ is potentially reducible and hence is automorphic by Proposition 3.3. Hence so is $\eta_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$, which is $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$ twisted by a finite character of $G_{\mathbb{Q}(\zeta_{2n})}$. Therefore $\sigma_{n,\ell,i}$ is automorphic, and so is its induction to $G_{\mathbb{Q}}$. This proves that $\rho_{n,\ell}^{new}$ is automorphic. □

Remark 7. When $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$ are assumed to be irreducible, we have seen from the above discussion that for each ℓ there exists a representation $\eta_{n,\ell,i}$ of G_{F_n} so that $\eta_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_n)}}$ is finitely projectively equivalent to $\sigma_{n,\ell,i}$. We claim that $\eta_{n,\ell,i}$ can be chosen to be part of a compatible system. More precisely, there exists a compatible system $\{E, S, \{Q_{\mathfrak{p}}(X)\}, \{\tilde{\eta}_{n,\lambda,i}\}\}$ of ℓ-adic Galois representation of G_{F_n} (see §2.4) so that for each ℓ there exists a prime λ of E over ℓ satisfying that $\tilde{\eta}_{n,\lambda,i}|_{G_{\mathbb{Q}(\zeta_n)}}$ is finitely projectively equivalent to $\sigma_{n,\ell,i}$. In other words, $\eta_{n,\ell,i}$ can be chosen to be some $\tilde{\eta}_{n,\lambda,i}$. Indeed, we start with the $\eta_{n,\ell,i}$ for each ℓ from the proof of Theorem 5.1 without knowing whether they are part of a compatible system. The argument of the proof (in particular, when we use Proposition 3.7) implies the existence of at least one large prime ℓ' so that $\eta_{n,\ell',i}$ is potentially automorphic. By Theorem 5.5.1 in [BGGT], there exists a compatible system $\{E, S, \{Q_{\mathfrak{p}}(X)\}, \{\tilde{\eta}_{n,\lambda,i}\}\}$ of ℓ-adic Galois representations of G_{F_n} so that for some prime λ' of E over ℓ' we have $\tilde{\eta}_{n,\lambda',i} = \eta_{n,\ell',i}$. To see that the system $\{\tilde{\eta}_{n,\lambda,i}\}$ has the desired property, it suffices to check that, for each prime λ of E over ℓ, $\tilde{\eta}_{n,\lambda,i}|_{G_{\mathbb{Q}(\zeta_n)}}$ is finitely projectively equivalent to $\sigma_{n,\ell,i}$. Let $\chi_{n,\ell',i}$ be the character with finite image so that $\eta_{n,\ell',i}|_{G_{\mathbb{Q}(\zeta_n)}} = \sigma_{n,\ell',i} \otimes \chi_{n,\ell',i}$. Since the image of $\chi_{n,\ell',i}$ in $\overline{\mathbb{Q}}_{\ell'}^{\times}$ is finite, it isomorphically embeds in $\overline{\mathbb{Q}}^{\times}$ so that it forms a compatible system of $G_{\mathbb{Q}(\zeta_n)}$. Denote by $\chi_{n,\ell,i}$ the same character with image viewed in $\overline{\mathbb{Q}}_\ell^\times$. We claim that, for all ℓ, $\tilde{\eta}_{n,\lambda,i}|_{G_{\mathbb{Q}(\zeta_n)}} = \sigma_{n,\ell,i} \otimes \chi_{n,\ell,i}$ for each λ above ℓ. As $\sigma_{n,\ell,i}$ is assumed to be irreducible, it suffices to check the trace of $\mathrm{Frob}_\mathfrak{p}$ on both sides for all primes $\mathfrak{p}$ of $\mathbb{Q}(\zeta_n)$ not dividing $n\ell$, for then the semi-simplification of $\tilde{\eta}_{n,\lambda,i}|_{G_{\mathbb{Q}(\zeta_n)}}$ and hence $\tilde{\eta}_{n,\lambda,i}|_{G_{\mathbb{Q}(\zeta_n)}}$ will be irreducible and equal to $\sigma_{n,\ell,i} \otimes \chi_{n,\ell,i}$. The claim then follows from the compatibility of three systems: $\{\tilde{\eta}_{n,\lambda,i}\}$, $\{\chi_{n,\ell,i}\}$, and $\{\sigma_{n,\ell,i}\}$ (by Corollary 5.4), and the fact that the equality holds for λ' and ℓ'.

5.7. $\tau_{n,\ell,i}$ **admits QM.** The concept of quaternion multiplication over a field was introduced in [AL^3, Definition 3.1.1] for 4-dimensional representations of $G_\mathbb{Q}$. We extend it to representations of finite index subgroups of $G_\mathbb{Q}$.

Definition 4. Let F be a number field. A finite-dimensional representation ρ of G_F is said to admit *quaternion multiplication* (*QM*) if there are two linear operators J_+ and J_- acting on the representation space of ρ such that

(1) $J_\pm^2 = -\,\mathrm{id}$ and $J_+J_- = -J_-J_+$ (so that $J_\pm$ generate the quaternion group Q_8).
(2) There exist two multiplicative characters $\chi_\pm$ of G_F of order ≤ 2 such that for any $g \in G_F$,
$$\rho(g) \circ J_\pm = \chi_\pm(g) J_\pm \circ \rho(g).$$

We say that ρ admits *QM over* L if $\chi_\pm$ are trivial on G_L.

Fix a choice of $1 \le i \le n-1$ coprime to n. We claim that the representation $\tau_{n,\ell,i} = \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{F_n}} \sigma_{n,\ell,i}$ of G_{F_n} studied in the previous subsection admits QM over $\mathbb{Q}(\zeta_{2n})$, generalizing the known results for $n = 3, 4, 6$ discussed in [AL^3]. To see this, recall the symmetry $A = \begin{pmatrix} -2 & -5 \\ 1 & 2 \end{pmatrix} \in \Gamma^0(5)$ mapping t_n to ζ_{2n}/t_n mentioned in Remark 6. It induces a map A on the model (5.4) of $\mathcal{E}_n$ sending (x, y, s) to $(1-x, 1/(1-xy), \zeta_{2n}(1-x)(1-y)x^2y/(s(1-xy)))$. The relation $\zeta A \zeta = A$ on $\mathcal{E}_n$ gives rise to the relation
$$\zeta^* A^* \zeta^* = A^*$$
satisfied by the operators ζ^* and A^* acting on the space of $\rho_{n,\ell}^{new}$. The operator ζ^* is defined over $\mathbb{Q}(\zeta_n)$ while A^* is defined over $\mathbb{Q}(\zeta_{2n})$. When n is odd, $\mathbb{Q}(\zeta_{2n}) = \mathbb{Q}(\zeta_n)$ is quadratic over the totally real subfield F_n of $\mathbb{Q}(\zeta_n)$. When n is even, $\mathbb{Q}(\zeta_{2n})$ is a biquadratic extension of $F_n = \mathbb{Q}(\cos \frac{2\pi}{n})$ with three quadratic intermediate fields: $F_{2n} = F_n(\cos \frac{2\pi}{2n})$, $\mathbb{Q}(\zeta_n) = F_n(\zeta_4 \sin \frac{2\pi}{n})$, and $F_n(\zeta_4 \sin \frac{2\pi}{2n})$. As finite abelian extensions of $\mathbb{Q}$, these three fields are characterized by the primes splitting completely in them, which are, respectively, $p \equiv \pm 1 \mod 2n$, $p \equiv 1 \mod n$, and $p \equiv 1, n-1 \mod 2n$. The primes p splitting completely in F_n are $\equiv \pm 1 \mod n$.

Lemma 5.9. *Let v be a prime of F_n dividing an odd prime $p \equiv \pm 1 \mod n$ so that $Nv = p$. Then on $\mathcal{E}_n$ we have* $\mathrm{Frob}_v A = \zeta^{(1-p)/2} A \,\mathrm{Frob}_v$ *and* $\mathrm{Frob}_v \zeta = \zeta^p \,\mathrm{Frob}_v$. *The actions of ζ^* and A^* on $\rho_{n,\ell}^{new}$ satisfy*
$$A^* \,\mathrm{Frob}_v = \mathrm{Frob}_v A^* (\zeta^*)^{(1-p)/2} \quad \textit{and} \quad \zeta^* \,\mathrm{Frob}_v = \mathrm{Frob}_v (\zeta^*)^p.$$
Here we retain the same notation for the Frobenius action on $\rho_{n,\ell}^{new}$.

Proof. We prove the identities on $\mathcal{E}_n$ by checking the actions of the maps on a point $(x, y, s) \in \mathcal{E}_n(\bar{\mathbb{Q}})$. The induced actions on $\rho_{n,\ell}^{new}$ satisfy the relations on $\mathcal{E}_n$

with reversed order because the operators act on a cohomology space. The first identity follows from

$$\begin{aligned}\mathrm{Frob}_v\, A(x,y,s) &= \mathrm{Frob}_v(1-x, 1/(1-xy), \zeta_{2n}(1-x)(1-y)x^2y/(s(1-xy))) \\ &= (1-\mathrm{Frob}_v(x), \frac{1}{1-\mathrm{Frob}_v(xy)}, \zeta_{2n}^p\,\mathrm{Frob}_v(\frac{(1-x)(1-y)x^2y}{s(1-xy)}))\end{aligned}$$

and

$$\begin{aligned}&\zeta^{(1-p)/2} A\,\mathrm{Frob}_v(x,y,s) \\ &\quad = \zeta^{(1-p)/2} A(\mathrm{Frob}_v(x), \mathrm{Frob}_v(y), \mathrm{Frob}_v(s)) \\ &\quad = \zeta^{(1-p)/2}(1-\mathrm{Frob}_v(x), \frac{1}{1-\mathrm{Frob}_v(xy)}, \zeta_{2n}\,\mathrm{Frob}_v(\frac{(1-x)(1-y)x^2y}{s(1-xy)})) \\ &\quad = (1-\mathrm{Frob}_v(x), 1/(1-\mathrm{Frob}_v(xy)), \zeta_{2n}^{p-1}\zeta_{2n}\,\mathrm{Frob}_v(\frac{(1-x)(1-y)x^2y}{s(1-xy)}));\end{aligned}$$

while the second identity results from $\zeta(x,y,s) = (x,y,\zeta_n^{-1}s)$ noted before:

$$\begin{aligned}\mathrm{Frob}_v\,\zeta(x,y,s) &= \mathrm{Frob}_v(x,y,\zeta_n^{-1}s) \\ &= (\mathrm{Frob}_v(x), \mathrm{Frob}_v(y), \zeta_n^{-p}\,\mathrm{Frob}_v(s)) = \zeta^p\,\mathrm{Frob}_v(x,y,s).\end{aligned}$$

□

To show that $\tau_{n,\ell,i}$ admits QM, consider the operators

$$B_+ := (1+(\zeta^*)^{-1})A^* \quad \text{and} \quad B_- := (1-(\zeta^*)^{-1})A^*$$

on the space of $\rho_{n,\ell}^{new}$. They leave invariant $\tau_{n,\ell,i} = \sigma_{n,\ell,i} \oplus \sigma_{n,\ell,n-i}$ since each summand is invariant under ζ^* and the two summands are swapped by A^*. It is straightforward to check, using the relations $\zeta^* A^* \zeta^* = A^*$ and $(A^*)^2 = -I$, that

$$B_+^2 = -(2+\zeta^*+(\zeta^*)^{-1}), \quad B_-^2 = -(2-\zeta^*-(\zeta^*)^{-1}), \quad \text{and} \quad B_+B_- = -B_-B_+.$$

Consequently on $\tau_{n,\ell,i}$, we have $B_+^2 = -(2+2\cos\frac{2i\pi}{n}) = -4\cos^2\frac{2i\pi}{2n}$ and $B_-^2 = -(2-2\cos\frac{2i\pi}{n}) = -4\sin^2\frac{2i\pi}{2n}$. Further,

$$B := B_+B_- = \zeta^* - (\zeta^*)^{-1}$$

on $\tau_{n,\ell,i}$ satisfies $B^2 = -4\sin^2\frac{2i\pi}{n}$.

Next we determine the commuting relations between these operators and G_{F_n}. As A^* and ζ^* commute with $G_{\mathbb{Q}(\zeta_{2n})}$, so do $B_\pm$ and B.

Proposition 5.10. (I) *When n is even, $\mathbb{Q}(\zeta_{2n})$ is a biquadratic extension of F_n. On $\rho_{n,\ell}^{new}$ we have*

(1) *B_+ commutes with $G_{F_{2n}}$, and $B_+\rho_{n,\ell}^{new}(g) = -\rho_{n,\ell}^{new}(g)B_+$ for $g \in G_{F_n} \smallsetminus G_{F_{2n}}$.*

(2) *B_- commutes with $G_{F_n(\zeta_4\sin\frac{2\pi}{2n})}$, and $B_-\rho_{n,\ell}^{new}(g) = -\rho_{n,\ell}^{new}(g)B_-$ for $g \in G_{F_n} \smallsetminus G_{F_n(\zeta_4\sin\frac{2\pi}{2n})}$.*

(3) *B commutes with $G_{\mathbb{Q}(\zeta_n)}$, and $B\rho_{n,\ell}^{new}(g) = -\rho_{n,\ell}^{new}(g)B$ for $g \in G_{F_n} \smallsetminus G_{\mathbb{Q}(\zeta_n)}$.*

(II) *When n is odd, $\mathbb{Q}(\zeta_{2n}) = \mathbb{Q}(\zeta_n)$ is a quadratic extension of F_n. On $\rho_{n,\ell}^{new}$ we have B_- and B commute with $G_{\mathbb{Q}(\zeta_n)}$ and anti-commute with elements in $G_{F_n} \smallsetminus G_{\mathbb{Q}(\zeta_n)}$.*

Therefore $\tau_{n,\ell,i}$ admits QM over $\mathbb{Q}(\zeta_{2n})$ with operators $J_+ := \frac{1}{2\sin\frac{2i\pi}{2n}}B_-$ and $J_- := \frac{1}{2\sin\frac{2i\pi}{n}}B$, and $\chi_\pm$ being quadratic characters of G_{F_n} with kernels $G_{F_n(\zeta_4\sin\frac{2\pi}{2n})}$ and $G_{\mathbb{Q}(\zeta_n)}$, respectively.

Proof. (I) For n even, observe that $\mathrm{Gal}(\mathbb{Q}(\zeta_{2n})/F_n) \cong G_{F_n}/G_{\mathbb{Q}(\zeta_{2n})}$ is a Klein four group consisting of Frob_v with places v of F_n above any prime $p \equiv 1, -1, n+1, n-1 \mod 2n$, respectively. This is because such primes p split completely in $\mathbb{Q}(\zeta_{2n})$, F_{2n}, $\mathbb{Q}(\zeta_n)$, and $F_n(\zeta_4 \sin \frac{2\pi}{2n})$, respectively, and these are the fixed fields of the corresponding Frob_v in $\mathbb{Q}(\zeta_{2n})$.

Now let v be a prime of F_n dividing an odd prime $p \equiv \pm 1 \mod n$, we examine the commuting relation between Frob_v and $B_\pm$ and B. Using Lemma 5.9, we get

$$\begin{aligned} B_\pm \mathrm{Frob}_v &= (1 \pm (\zeta^{-1})^*)A^* \mathrm{Frob}_v = (1 \pm (\zeta^{-1})^*) \mathrm{Frob}_v A^*(\zeta^*)^{(1-p)/2} \\ &= \mathrm{Frob}_v(1 \pm (\zeta^*)^{-p})A^*(\zeta^*)^{(1-p)/2} = \mathrm{Frob}_v(1 \pm (\zeta^*)^{-p})(\zeta^*)^{(p-1)/2}A^*. \end{aligned}$$

For $p \equiv 1 \mod 2n$, the place v of F_n splits completely in $\mathbb{Q}(\zeta_{2n})$. As observed above, $B_\pm$ and B commute with such Frob_v. For $p \equiv -1 \mod 2n$, we have

$$B_+ \mathrm{Frob}_v = \mathrm{Frob}_v(1 + \zeta^*)(\zeta^*)^{-1}A^* = \mathrm{Frob}_v B_+$$

and for $p \equiv n \pm 1 \mod 2n$ it is straightforward to verify $B_+ \mathrm{Frob}_v = -\mathrm{Frob}_v B_+$. This proves (1).

For (2) we find for v above $p \equiv n-1 \mod 2n$,

$$B_- \mathrm{Frob}_v = \mathrm{Frob}_v(1 - (\zeta^*))(\zeta^*)^{n/2-1}A^* = -\mathrm{Frob}_v((\zeta^*)^{-1} - 1) = \mathrm{Frob}_v B_-$$

and one checks that for v above $p \equiv -1$ and $n+1 \mod 2n$ we have $B_- \mathrm{Frob}_v = -\mathrm{Frob}_v B_-$. This proves (2).

For (3) we note that

$$B \,\mathrm{Frob}_v = (\zeta^* - (\zeta^*)^{-1}) \mathrm{Frob}_v = \mathrm{Frob}_v((\zeta^*)^p - (\zeta^*)^{-p}).$$

Thus for $p \equiv 1 \mod n$ this gives $B \,\mathrm{Frob}_v = \mathrm{Frob}_v B$ and for $p \equiv -1 \mod n$ this gives $B\,\mathrm{Frob}_v = -\mathrm{Frob}_v B$, as asserted in (3).

(II) When n is odd, $\mathbb{Q}(\zeta_{2n}) = \mathbb{Q}(\zeta_n)$ so that ζ^* and A^* commute with $G_{\mathbb{Q}(\zeta_n)}$, and so do $B_\pm$ and B. Since $F_{2n} = F_n$ and $F_n(\zeta_4 \sin \frac{2\pi}{2n}) = \mathbb{Q}(\zeta_n)$ in this case, the above computation shows that B_- and B commute with $G_{\mathbb{Q}(\zeta_n)}$ and anti-commute with elements in G_{F_n} but outside $G_{\mathbb{Q}(\zeta_n)}$, while B_+ commutes with G_{F_n}. This completes the proof of the proposition. $\square$

Now we discuss the situation where the automorphy of $\rho_{n,\ell}^{new} = \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_\mathbb{Q}} \sigma_{n,\ell,i}$ for $(i,n) = 1$ is not yet known. Then $\sigma_{n,\ell,i}$ is strongly irreducible by Proposition 5.8. Since $\tau_{n,\ell,i} = \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{F_n}} \sigma_{n,\ell,i}$ admits QM over $\mathbb{Q}(\zeta_{2n})$ (Proposition 5.10), following the same argument as the proof of Theorem 3.2.1 in [AL^3], one obtains a finite character ξ_n of $G_{\mathbb{Q}(\zeta_n)}$ such that $\sigma_{n,\ell,i} \otimes \xi_n$ extends to a degree-2 representation $\eta_{n,\ell,i}$ of G_{F_n} and $\tau_{n,\ell,i} = \eta_{n,\ell,i} \otimes \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{F_n}} \xi_n^{-1}$. To handle the case that $\tau_{n,\ell,i}$ is irreducible, Case (ii) of the proof of Theorem 5.1 used Clifford theory (Theorem 2.3) to conclude $\tau_{n,\ell,i} = \eta_{n,\ell,i} \otimes \gamma_{n,\ell,i}$ for a degree-2 representation $\eta_{n,\ell,i}$ of G_{F_n} whose restriction to $G_{\mathbb{Q}(\zeta_{2n})}$ differs from $\sigma_{n,\ell,i}|_{G_{\mathbb{Q}(\zeta_{2n})}}$ by a finite character, and a degree-2 representation $\gamma_{n,\ell,i}$ of G_{F_n} with finite image. Here using the QM structure, we gain the information that $\gamma_{n,\ell,i}$ can be chosen to be the representation induced from a finite character of $G_{\mathbb{Q}(\zeta_n)}$, and thus is automorphic. When $\tau_{n,\ell,i}$ is reducible, Case (i) of the proof of Theorem 5.1 shows that we can also write $\tau_{n,\ell,i} = \eta_{n,\ell,i} \otimes \gamma_{n,\ell,i}$ with $\gamma_{n,\ell,i}$ being the sum of two finite characters of G_{F_n}, in other words, $\mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{F_n}} \xi_n^{-1}$ decomposes into the sum of two finite characters, which is also automorphic. The same argument as in Remark 7 shows that we may assume $\eta_{n,\ell,i}$ to be part of a compatible system.

We summarize the conclusion of this section in the following remark.

Remark 8. For $n \geq 3$, $(i,n) = 1$ and a prime ℓ, the representation $\tau_{n,\ell,i}$ admits QM over $\mathbb{Q}(\zeta_{2n})$. By Theorem 3.1 the degree-2 representation $\eta_{n,\ell,i}$ of G_{F_n} occurring in $\tau_{n,\ell,i}$ as above is totally odd, with Hodge-Tate weights 0 and -2, and potentially automorphic. Moreover, it is automorphic if $F_n = \mathbb{Q}$ or it is potentially reducible. Further $\eta_{n,\ell,i}$ can be chosen to be part of a compatible system and $\tau_{n,\ell,i}$ is automorphic if and only if $\eta_{n,\ell,i}$ is. Therefore the automorphy of $\eta_{n,\ell,i}$ for all $(n,i) = 1$ will imply the automorphy of $\rho_{n,\ell}^{new}$ by automorphic induction.

5.8. **Remarks on other families of Scholl representations attached to noncongruence subgroups.** The noncongruence groups Γ_n for $n \neq 5$ by construction are finite index normal subgroups of $\Gamma^1(5)$. The group $\Gamma^1(5)$ is one of the six isomorphism classes of torsion-free index-12 subgroups in $PSL_2(\mathbb{Z})$. In [FHLRV], the authors constructed similar noncongruence subgroups from other torsion-free index-12 subgroups, such as $\Gamma_1(6)$. Following Beauville, the authors in [FHLRV] use the equation $(xy + yx + zx)(x + y + z) = txyz/9$ for the universal family of elliptic curves with an order 6 torsion point (see Table 6 of [FHLRV]) where $t = \frac{\eta(6z)^4\eta(z)^8}{\eta(3z)^8\eta(2z)^4}$ is a Hauptmodul for $\Gamma_1(6)$. This plays the same role as (5.3) in the $\Gamma^1(5)$ case. When one considers cyclic cover of the modular curve of $\Gamma_1(6)$ by replacing t by $t_n = \sqrt[n]{t}$ and denotes by $\tilde{\Gamma}_n$ the corresponding index-n subgroup of $\Gamma_1(6)$, then $S_3(\tilde{\Gamma}_n)$ is of dimension $n-1$ and the modular curve $X_{\tilde{\Gamma}_n}$ and the universal elliptic curve over it admit the automorphism $\zeta : t_n \mapsto \zeta_n^{-1}t_n$. Because of ζ the Scholl representations associated with $S_3(\tilde{\Gamma}_n)$, when restricted to $G_{\mathbb{Q}(\zeta_n)}$, also decompose into a sum of degree-2 factors $\sigma_{n,\ell,i}$ for $1 \leq i < n$. Using the same argument, one has $\sigma_{n,\ell,i}^c \cong \sigma_{n,\ell,n-i}$ and they are swapped by the W_3 operator of $\Gamma_1(6)$ which plays the role of A for $\Gamma^1(5)$. On the universal elliptic curve, W_3 gives rise to an isogeny map; see §5.1 of [FHLRV]. So, one can draw the same (potential) automorphy conclusion for the corresponding Scholl representations. It is worth pointing out that all cusps of $\Gamma_1(6)$ are defined over $\mathbb{Q}$, and any cusp can be sent to ∞ via one of the Atkin-Lenher involutions (see Table 12 of [FHLRV] for the corresponding linear transformation on t). Consequently one can construct other infinite families of cyclic subgroups of $\Gamma_1(6)$ ramified only at two cusps which give rise to Galois representations of $G_{\mathbb{Q}}$ for which similar (potential) automorphy conclusions can be drawn.

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Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Email address: `wli@math.psu.edu`

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

Email address: `tongliu@math.purdue.edu`

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803

Email address: `llong@lsu.edu`