LATTICES IN FILTERED (φ, N)-MODULES

TONG LIU

ABSTRACT. Let p be a prime. We construct and study integral and torsion invariants, such as integral and torsion Weil-Deligne representations, associated to potentially semi-stable representations and torsion potentially semi-stable representations respectively. As applications, we prove the compatibility between local Langlands correspondence and Fontaine's construction for Galois representations attached to Hilbert modular forms, and Néron-Ogg-Shafarevich criterion of finite level for potentially semi-stable representations.

CONTENTS

1. Introduction 1
2. Lattices in filtered (φ, N, Γ)-module $D_{\mathrm{st}}(V)$ 3
2.1. The main result in §2 3
2.2. $(\varphi, \hat{G})$-modules 5
2.3. Construction of M_{st} 10
2.4. Stability of monodromy 12
2.5. Some properties of M_{st} 14
3. p-adic Hodge data of torsion representations 17
3.1. Construction of M_{st} for torsion representations 17
3.2. Torsion Kisin modules 19
3.3. Compatibility of Γ and N 23
3.4. Representations with coefficients 24
4. Applications 25
4.1. Néron-Ogg-Shafarevich criterion of finite level 25
4.2. Local Langlands compatibility of Hilbert modular forms 26
References 31

1. INTRODUCTION

Let k be a perfect field of characteristic p, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[1/p]$, K/K_0 a finite totally ramified extension, $G_K := \mathrm{Gal}(\overline{K}/K)$ and V a potentially semi-stable p-adic representation of G_K. In [Fon94b], Fontaine

1991 *Mathematics Subject Classification.* Primary 14F30,14L05.
Key words and phrases. Potentially semi-stable representations, filtered (φ, N)-modules.

associated a filtered (φ, N, G_K)-module $D^*_{\mathrm{st}}(V)$ [1] to V. Many important invariants of V can be read from $D^*_{\mathrm{st}}(V)$, for example, the Weil-Deligne representation attached to V if K is a finite extension over $\mathbb{Q}_p$. On the other hand, since there is always an integral (resp. p^n-torsion) structure attached to V, i.e., a G_K-stable $\mathbb{Z}_p$-lattice T (resp. T/p^nT) in V, it is natural to ask if there exists a corresponding integral (resp. p^n-torsion) structure attached to T (resp. T/p^nT). If $K = K_0$ and V is crystalline with Hodge-Tate weights inside $\{0, \dots, p-2\}$, one can make such a construction via Fontaine-Laffaille theory in [FL82]. The aim of this paper is to extend some parts of this construction to the setting of potentially semi-stable representations without restriction of ramification of K and Hodge-Tate weight.

More precisely, we fix a finite totally ramified Galois extension K'/K and consider the category $\mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Q}_p}$ whose objects are representations of G_K which are semi-stable over K' with Hodge-Tate weights in $\{0, \dots, r\}$. Let $V \in \mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Q}_p}$. For any G_K-stable $\mathbb{Z}_p$-lattice $T \subset V$, we construct a lattice $M_{\mathrm{st}}(T)$ inside the filtered (φ, N, G_K)-module $D_{\mathrm{st}}(V) := D^*_{\mathrm{st}}(V^\vee)$ ($V^\vee$ is the $\mathbb{Q}_p$-dual of V) via the theory of $(\varphi, \hat{G})$-modules in [Liu09b]. Lattice here means finite free $W(k)$-module inside $D_{\mathrm{st}}(V)$ which is stable under φ, N and the G_K-action, see Definition 2.1.2 for details. Moreover, the contravariant functor M_{st} allows us to construct similar invariants for torsion potentially semi-stable representations as the following: let $\mathrm{Rep}^{\mathrm{pst},r}_{\mathrm{tor}}$ be the category whose objects are p-power torsion representations T such that there exists a pair of G_K-stable $\mathbb{Z}_p$-lattices $L \overset{\mathcal{L}}{\hookrightarrow} L'$ inside a potentially semi-stable representation $V \in \mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Q}_p}$ satisfying $T \simeq L'/L$. We set $M_{\mathrm{st},\mathcal{L}}(T) := M_{\mathrm{st}}(L)/M_{\mathrm{st}}(L')$. So $M_{\mathrm{st},\mathcal{L}}(T)$ has structures of φ, N and G_K-action induced from those on $M_{\mathrm{st}}(L)$. In general, $M_{\mathrm{st},\mathcal{L}}(T)$ not only depends on T but also on the choice of lattices $L \overset{\mathcal{L}}{\hookrightarrow} L'$. However we prove that there exists a constant $\mathfrak{c}$ only depending on r and the ramification index $e = e(K'/K_0)$ such that the construction of $M_{\mathrm{st},\mathcal{L}}(T)$ is "independent on" the choice of $\mathcal{L}$ up to a $p^{\mathfrak{c}}$-power (see Theorem 3.1.1 for the precise statement).

In the last section, we provide two applications based on the above constructions. The first application is to extend Néron-Ogg-Shafarevich criterion to finite level as in [Liu07a]:

Theorem 1.0.1. *Let V be a potentially semi-stable representation of G_K with Hodge-Tate weights in $\{0, \dots, r\}$, and $T \subset V$ a G_K-stable $\mathbb{Z}_p$-lattice. There exists a constant α depending on the dimension $d = \dim_{\mathbb{Q}_p}(V)$, the absolute ramification index $\tilde{e} = [K : K_0]$ and r such that V is semi-stable over K if and only if there exist G_K-stable $\mathbb{Z}_p$-lattices $L' \subset L$ in a semi-stable representation W of G_K with Hodge-Tate weights in $\{0, \dots, r\}$ satisfying $T/p^\alpha T \simeq L/L'$.*

The second application is

Theorem 1.0.2. *Let F be a totally real field and π a Hilbert eigenform of weight $\underline{k} = (k_1, \dots, k_i)$ with $k_i \geq 2$, integers all have the same parity. Let ρ_π denote the 2-dimensional p-adic Galois representation of $G_F := \mathrm{Gal}(\overline{F}/F)$ attached to π.*

If $\mathfrak{q}|p$ is a prime of F, and $G_{F_\mathfrak{q}}$ denote a decomposition group at $\mathfrak{q}$. Then $\rho_\pi|_{G_{F_\mathfrak{q}}}$ is potentially semi-stable with p-adic Hodge type corresponding to the weight $\underline{k}$.

---

[1] Our conventions are slightly different from those in [Fon94b], see Convention 2.1.1 for details.

Moreover, the Weil-Deligne representation attached to $\rho_\pi|_{G_{F_\mathfrak{q}}}$ via Fontaine's construction corresponds to the local factor $\pi_\mathfrak{q}$ of π via local Langlands correspondence.

The condition on p-adic Hodge type means the following: We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$. If ρ_π is defined over a finite extension $E/\mathbb{Q}_p$ and $m = \max_i(k_i)$, then for any $\mathfrak{q}|p$ there is a graded $E \otimes_{\mathbb{Q}_p} F_\mathfrak{q}$-module $D_\mathfrak{q}$ attached to the Hodge-Tate representation $\rho_\pi|_{G_{F_\mathfrak{q}}}$. If $j : F_\mathfrak{q} \to \overline{\mathbb{Q}}_p$ is an embedding, then $D_j = D_\mathfrak{q} \otimes_{F_\mathfrak{q}, j} \overline{\mathbb{Q}}_p$ is non-zero in the degrees $(m - k_{i(j)})/2$ and $(m + k_{i(j)} - 2)/2$, where $k_{i(j)}$ is the weight determined by the embedding $j|_F : F \to \overline{\mathbb{Q}}_p \simeq \mathbb{C}$.

Many cases of the above theorem have been known: When $[F : \mathbb{Q}]$ is odd or $[F : \mathbb{Q}]$ is even and π is a discrete series at some finite place w, this is proved by Saito ([Sai]) based on the construction of ρ_π in this case by Carayol ([Car86]). Under the same hypothesis or when $[F : \mathbb{Q}]$ is even and some k_i is strictly larger than 2, Blasius and Rogawski proved that $\rho_\pi|_{G_{F_\mathfrak{q}}}$ is potentially semi-stable with p-adic Hodge type corresponding to the weight $\underline{k}$, and when additionally p is sufficiently large they showed the fully conclusion of the theorem holds in [BR93]. Partial results can also be found in [Tay95] and [Bre99]. For those ρ_π which are residually absolutely irreducible, Kisin proved the theorem in [Kis08]. When this paper is nearly complete, the author has learned that Skinner has also proved the theorem ([Ski]). The author remark that Skinner use a very different approach from ours.

Acknowledgment: It is a pleasure to thank Don Blasius, Brian Conrad, Mark Kisin, Takeshi Saito, Christopher Skinner for very useful conversations and correspondences during the preparation of this paper. Here I would like to specially thank Mark Kisin for illuminating me the idea for the proof of Theorem 1.0.2.

2. Lattices in filtered (φ, N, Γ)-module $D_{\mathrm{st}}(V)$

2.1. **The main result in §2.** Recall k is a perfect field of characteristic p, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[\frac{1}{p}]$, K/K_0 a finite totally ramified extension and $G_K := \mathrm{Gal}(\overline{K}/K)$. Throughout §2 and §3, we fix a totally ramified Galois extension K' over K and a uniformiser $\pi \in K'$ with Eisenstein polynomial $E(u) \in W(k)[u]$. We write $G := G_{K'} = \mathrm{Gal}(\overline{K}/K')$, $\Gamma := \mathrm{Gal}(K'/K)$ and $e = [K' : K_0]$.

Following [Fon94d], recall that a p-adic representation V of G_K is called *semi-stable* over K if

$$\dim_{K_0}(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \dim_{\mathbb{Q}_p}(V).$$

A p-adic representation V of G_K is called *potentially semi-stable* if there exists a finite extension $\tilde{K}/K$ such that $V|_{\mathrm{Gal}(\overline{K}/\tilde{K})}$ is semi-stable over $\tilde{K}$. Throughout §2 and §3, we always make the following assumption:

any potentially semi-stable representation of G_K considered here is semi-stable over K'.

We denote by $\mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}$ the category of potentially semi-stable representations of G_K which satisfy the above assumption.

Following [Fon94d], a *filtered (φ, N, Γ)-module* is a finite dimensional K_0-vector space D endowed with:

- a Frobenius semi-linear injection: $\varphi : D \to D$.
- a $W(k)$-linear map $N : D \to D$ such that $N\varphi = p\varphi N$.

- a decreasing filtration $(\mathrm{Fil}^i D_{K'})_{i\in\mathbb{Z}}$ on $D_{K'} := K' \otimes_{K_0} D$ by K'-vector spaces such that $\mathrm{Fil}^i D_{K'} = D_{K'}$ for $i \ll 0$ and $\mathrm{Fil}^i D_{K'} = 0$ for $i \gg 0$.
- a K_0-linear Γ-action on D such that Γ commutes with φ and N; if we extend Γ semi-linearly to $D_{K'}$ then Γ preserves $\mathrm{Fil}^i D_{K'}$, i.e., for any $\gamma \in \Gamma$ and $i \in \mathbb{Z}$, $\gamma(\mathrm{Fil}^i D_{K'}) \subset \mathrm{Fil}^i D_{K'}$.

Morphisms between filtered (φ, N, Γ)-modules are K_0-linear maps preserving all structures. By [CF00] and [Fon94c], the functor [2]

$$D^*_{\mathrm{st}}(V): \quad V \mapsto (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_{K'}}$$

induces an equivalence between the category $\mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}$ and the category of *weakly admissible* filtered (φ, N, Γ)-modules. See [CF00] for the definition of weakly admissibility.

In the sequel, we will instead use the contravariant functor $D_{\mathrm{st}}(V) := D^*_{\mathrm{st}}(V^\vee)$, where $V^\vee$ is the dual representation of V, because contravariant functors are more convenient in the integral theory. So let us remind the readers the problem of notations.

Convention 2.1.1. Here we use slightly different conventions from those in [CF00] and [Fon94d], where D_{st} defined here should be denoted by $D^*_{\mathrm{st},K'}$ if we follow the traditional conventions. But *contravariant* functors instead of covariant functors dominate this paper and K' is always fixed in §2 and §3. So we decide to use the simplified notation D_{st}. For any finite $\mathbb{Z}_p$-module ($\mathbb{Q}_p$-module) V, we use $V^\vee$ to denote its $\mathbb{Z}_p$-dual ($\mathbb{Q}_p$-dual). In particular, if V is killed by some p-power, $V^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(V, \mathbb{Q}_p/\mathbb{Z}_p)$. We will define p-adic Hodge structures such as Frobenius, monodromy on many different rings and modules. To distinguish them, we sometime add subscripts to indicate over which those structures are defined. For example, $\varphi_{\mathfrak{M}}$ is the Frobenius defined on $\mathfrak{M}$. We always drop these subscripts if no confusions arise. Throughout this paper, we reserve φ and N for various types of Frobenius and monodromy respectively. Finally, we denote $\gamma_i(x)$ the standard divided power $\frac{x^i}{i!}$, $\mathrm{M}_{d\times d}(A)$ the ring of $d \times d$-matrices with coefficients in ring A and Id the identity map.

The aim of this section is to develop an integral theory of D_{st}. First, we define integral structure in filtered (φ, N, Γ)-module.

Definition 2.1.2. Let D be a filtered (φ, N, Γ)-module. A *lattice* M in D is a $W(k)$-submodule of D such that

- M is $W(k)$-finite free and $M[\frac{1}{p}] \overset{\sim}{\to} D$
- M is stable under φ, N and Γ, i.e., $\varphi(M) \subset M$, $N(M) \subset M$ and $\gamma(M) \subset M$ for any $\gamma \in \Gamma$.

Let M_i be a lattice in a filtered (φ, N, Γ)-module D_i for $i = 1, 2$. A morphism $f : M_1 \to M_2$ between two lattices is a $W(k)$-linear map such that $f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a morphism of filtered (φ, N, Γ)-modules. That is, $f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is still required to preserve filtration.*

Throughout §2 and §3, we fix a positive integer $r > 0$. Let $\mathrm{L}^r(\varphi, N, \Gamma)$ denote the category of lattices in filtered (φ, N, Γ)-modules satisfying $\mathrm{Fil}^0 D_{K'} = D_{K'}$

[2] Note our notations are slightly different from those in [CF00].

and $\mathrm{Fil}^{r+1}_{K'} D_{K'} = 0$. We denote by $\mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Q}_p}$ the full subcategory of $\mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}$ whose representations have Hodge-Tate weights in $\{0, \dots, r\}$, and by $\mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Z}_p}$ the category of G_K-stable $\mathbb{Z}_p$-lattices in representations which are in $\mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Q}_p}$.

Now we can state our main result in §2:

Theorem 2.1.3. *There exists a left exact and faithful functor M_{st} from the category $\mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Z}_p}$ to the category $\mathrm{L}^r(\varphi, N, \Gamma)$. Moreover, let $M_{\mathrm{st}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ denote the functor M_{st} associated to the isogeny categories. Then there is a natural isomorphism between $M_{\mathrm{st}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and D_{st}. If $er < p-1$ then M_{st} is exact and fully faithful.*

If $er \geq p-1$ then M_{st} in general is neither full nor exact. See Example 2.5.5 and Example 2.5.6.

Remark 2.1.4. It is important to describe objects in the essential image of M_{st}. But so far we do not know how to do it, though we guess one need impose some suitable conditions on filtration, just as those in Fontaine-Laffaille theory [FL82]. This is one reason that we do not put any restriction on filtration in Definition 2.1.2.

2.2. $(\varphi, \hat{G})$**-modules.** We first review the theory of $(\varphi, \hat{G})$-modules from [Liu09b] to manipulate the lattices in semi-stable representations. We only deal with the integral theory in this section, and the torsion theory will be discussed in §3.

Recall the fixed uniformiser $\pi \in K'$ with Eisenstein polynomial $E(u)$. Put $\mathfrak{S} := W(k)[\![u]\!]$. $\mathfrak{S}$ is equipped with a Frobenius endomorphism φ via $u \mapsto u^p$ and the natural Frobenius on $W(k)$. A *φ-module* (over $\mathfrak{S}$) is an $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$. A morphism between two objects $(\mathfrak{M}_1, \varphi_1)$, $(\mathfrak{M}_2, \varphi_2)$ is an $\mathfrak{S}$-linear morphism compatible with the φ_i. Denote by $'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$ the category of *φ-modules of height*[3] *r*, in the sense that $\mathfrak{M}$ is $\mathfrak{S}$-finite type and the cokernel of φ^* is killed by $E(u)^r$, where φ^* is the $\mathfrak{S}$-linear map $1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$. By definition, a *finite free Kisin module* (of height r) is a φ-module (of height r) $\mathfrak{M}$ such that $\mathfrak{M}$ is finite $\mathfrak{S}$-free. We write $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ for the category of finite free Kisin modules of height r.

We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. Write $S_{K_0} := S[\frac{1}{p}]$. There is a unique map (Frobenius) $\varphi : S \to S$ which extends the Frobenius on $\mathfrak{S}$. We write N_S for the K_0-linear derivation on S_{K_0} such that $N_S(u) = -u$.

Let $R = \varprojlim \mathcal{O}_{\overline{K}}/p$ where the transition maps are given by Frobenius. By the universal property of the Witt vectors $W(R)$ of R, there is a unique surjective projection map $\theta : W(R) \to \widehat{\mathcal{O}}_{\overline{K}}$ to the p-adic completion of $\mathcal{O}_{\overline{K}}$, which lifts the projection $R \to \mathcal{O}_{\overline{K}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$ with respect to $\mathrm{Ker}(\theta)$. Let $\pi_n \in \overline{K}$ be a p^n-th root of π, such that $(\pi_{n+1})^p = \pi_n$; write $\underline{\pi} = (\pi_n)_{n \geq 0} \in R$ and let $[\underline{\pi}] \in W(R)$ be the Techmüller representative. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R) \subset A_{\mathrm{cris}}$ by the map $u \mapsto [\underline{\pi}]$. This embedding extends to embeddings $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$ which are compatible with Frobenious endomorphisms. As usual, we write $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$ and $B^+_{\mathrm{st}} = B^+_{\mathrm{cris}}[\mathfrak{u}]$ with $\mathfrak{u} := \log([\underline{\pi}])$. We write N for the B^+_{cris}-linear derivation on B^+_{st} by setting $N(\mathfrak{u}) = 1$, and denote by B^+_{dR}

[3]Strictly speaking, it should be called $E(u)$-height because it depends on the choice of $E(u)$. Since we always fix a $E(u)$, there is no confusion here to drop $E(u)$.

the $\mathrm{Ker}(\theta)$-adic completion of $W(R)[1/p]$. For any subring $A \subset B^+_{\mathrm{dR}}$, we define filtration on A by $\mathrm{Fil}^i A = A \cap (\mathrm{Ker}(\theta))^i B^+_{\mathrm{dR}} = A \cap (E(u)^i) B^+_{\mathrm{dR}}$.

Let $K'_\infty := \bigcup_{n=0}^{\infty} K'(\pi_n)$ and $\hat{K}'$ its Galois closure over K'. Then $\hat{K}' = \bigcup_{n=1}^{\infty} K'_\infty(\zeta_{p^n})$ with ζ_{p^n} a primitive p^n-th root of unity. Write $G_\infty := \mathrm{Gal}(\overline{K}/K'_\infty)$, $K'_{p^\infty} = \bigcup_{n=1}^{\infty} K'(\zeta_{p^n})$, $G_{p^\infty} := \mathrm{Gal}(\hat{K}'/K'_{p^\infty})$, $H_{K'} := \mathrm{Gal}(\hat{K}'/K'_\infty)$ and $\hat{G} := \mathrm{Gal}(\hat{K}'/K')$.

For any $g \in G_K$, write $\underline{\epsilon}(g) := \frac{g(\pi)}{\pi}$, which is a cocycle from G_K to R^*. Fix a choice of *primitive* p^i-root of unity ζ_{p^i} for $i \geq 0$ and set $\underline{\epsilon} := (\zeta_{p^i})_{i\geq 0} \in R$ and $t := \log([\underline{\epsilon}]) \in A_{\mathrm{cris}}$. We see that $g(t) = \chi(g)t$ with χ the p-adic cyclotomic character, and there exists an $\alpha(g) \in \mathbb{Z}_p$ such that $\log([\underline{\epsilon}(g)]) = \alpha(g)t$.

As a subring of A_{cris}, S is not stable under the action of G, though S is fixed by G_∞. Define a subring inside B^+_{cris}:

$$\mathcal{R}_{K_0} := \left\{ x = \sum_{i=0}^{\infty} f_i t^{\{i\}}, f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to +\infty \right\},$$

where $t^{\{i\}} = \frac{t^i}{p^{\tilde{q}(i)}\tilde{q}(i)!}$ and $\tilde{q}(i)$ satisfies $i = \tilde{q}(i)(p-1) + r(i)$ with $0 \leq r(i) < p-1$. Define $\widehat{\mathcal{R}} := W(R) \cap \mathcal{R}_{K_0}$. One can show that $\mathcal{R}_{K_0}$ and $\widehat{\mathcal{R}}$ are stable under the G-action and the G-action factors through $\hat{G}$ (see [Liu09b] §2.2). R is a valuation ring. Write $v_R(\cdot)$ for the valuation and let $I_+R = \{x \in R | v_R(x) > 0\}$ be the maximal ideal of R. We have an exact sequences

$$0 \to W(I_+R) \to W(R) \xrightarrow{\nu} W(\bar{k}) \to 0.$$

One can naturally extend ν to $\nu : B^+_{\mathrm{cris}} \to W(\bar{k})[\frac{1}{p}]$ (see the proof of Lemma 2.2.1 in [Liu09b]). For any subring A of B^+_{cris}, we write $I_+A = \mathrm{Ker}(\nu) \cap A$ and $I_+ := I_+\widehat{\mathcal{R}}$. It is not hard to see that $I_+\mathfrak{S} = u\mathfrak{S}$ and $I_+S = \{x \in S | x = \sum_{i=1}^{\infty} a_i \frac{u^i}{q(i)!},\ a_i \in W(k)\}$, where $q(i)$ satisfies $i = eq(i) + r(i)$ with $0 \leq r(i) < e$. By Lemma 2.2.1 in [Liu09b], one have $\widehat{\mathcal{R}}/I_+ \simeq \mathfrak{S}/u\mathfrak{S} \simeq S/I_+S = W(k)$.

Following [Liu09b], a *finite free $(\varphi, \hat{G})$-module of height r* is a triple $(\mathfrak{M}, \varphi, \hat{G})$ where

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a finite free Kisin module of height r;
(2) $\hat{G}$ is a $\widehat{\mathcal{R}}$-semi-linear $\hat{G}$-action on $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$;
(3) $\hat{G}$ commutes with $\varphi_{\hat{\mathfrak{M}}}$ on $\hat{\mathfrak{M}}$, *i.e.*, for any $g \in \hat{G}$, $g\varphi_{\hat{\mathfrak{M}}} = \varphi_{\hat{\mathfrak{M}}} g$;
(4) regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule in $\hat{\mathfrak{M}}$, then $\mathfrak{M} \subset \hat{\mathfrak{M}}^{H_K}$;
(5) $\hat{G}$ acts on $W(k)$-module $M := \hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}} \simeq \mathfrak{M}/u\mathfrak{M}$ trivially.

A morphism between two finite free $(\varphi, \hat{G})$-modules is a morphism in $\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$ that commutes with $\hat{G}$-action on $\hat{\mathfrak{M}}$'s. We denote by $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$ the category of finite free $(\varphi, \hat{G})$-modules of height r. For a finite free $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ [4], we can associate a $\mathbb{Z}_p[G]$-module:

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, W(R)), \quad (2.2.1)$$

[4] We also (always) use $\hat{\mathfrak{M}}$ to denote $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. This is a (harmless) abuse of notations.

where G acts on $\hat{T}(\hat{\mathfrak{M}})$ via $g(f)(x) = g(f(g^{-1}(x)))$ for any $g \in G$ and $f \in \hat{T}(\hat{\mathfrak{M}})$.

The theory of $(\varphi, \hat{G})$-modules is built on Kisin's theory ([Kis06]) on the classification of G_∞-stable $\mathbb{Z}_p$-lattices via Kisin modules. We refer [Kis06] and [Liu09b] for the more details of Kisin's theory. Recall $\mathfrak{S}$ embeds to $W(R)$ via $u \mapsto [\underline{\pi}]$. Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic closure of $\mathfrak{S}[1/u]$ in $W(\mathrm{Frac}R)$. Define $\mathcal{E} = \mathrm{Frac}\mathcal{O}_{\mathcal{E}}$ and $\widehat{\mathcal{E}}^{\mathrm{ur}}$ the p-adic completion of the maximal (algebraic) unramified extension of $\mathcal{E}$ in $W(\mathrm{Frac}R)[1/p]$. Denote $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$ its ring of integers and put $\mathfrak{S}^{\mathrm{ur}} = W(R) \cap \mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$. $\mathfrak{S}^{\mathrm{ur}}$ is a subring of $W(R)$ stable under the G_∞-action and Frobenius. Denote by $\mathrm{Rep}(G_\infty)$ the category of $\mathbb{Z}_p[G_\infty]$-modules of finite $\mathbb{Z}_p$-type. For any $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$, we can associate a finite $\mathbb{Z}_p$-free object in $\mathrm{Rep}(G_\infty)$ via

$$T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}).$$

By Example 2.3.5 in [Liu07b], there exists an element $\mathfrak{t} \in W(R)$ such that $\mathfrak{t} \bmod p \neq 0$, $\varphi(\mathfrak{t}) = c_0^{-1} E(u)\mathfrak{t}$, where $c_0 p$ is the constant term of $E(u)$. Such $\mathfrak{t}$ is unique up to units in $\mathbb{Z}_p$. Example 5.3.3 in [Liu07b] showed that we can select $\mathfrak{t}$ such that $t = c\varphi(\mathfrak{t})$ with $c = \prod_{n=0}^{\infty} \varphi(\frac{\varphi^n(c_0^{-1}E(u))}{p})$ a unit in S. So throughout this paper we fix the selection of $\mathfrak{t}$ such that $t = c\varphi(\mathfrak{t})$. The main result in [Liu09b] (cf. Theorem 2.3.1 and Proposition 3.1.3) is the following:

Theorem 2.2.1 ([Liu09b])**.** (1) *$\hat{T}$ induces an anti-equivalence between the category of finite free $(\varphi, \hat{G})$-modules of height r and the category of G-stable $\mathbb{Z}_p$-lattices in semi-stable representations of G with Hodge-Tate weights in $\{0, \dots, r\}$.*

(2) *$\hat{T}$ induces a natural $W(R)$-linear injection*

$$\hat{\iota}: \ W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \longrightarrow \hat{T}^{\vee}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R), \tag{2.2.2}$$

such that $\hat{\iota}$ is compatible with Frobenius and G-actions on both sides. Moreover, $(\varphi(\mathfrak{t}))^r(\hat{T}^{\vee}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R)) \subset \hat{\iota}(W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}})$.

(3) *There exists a natural isomorphism $T_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}(\hat{\mathfrak{M}})$ of $\mathbb{Z}_p[G_\infty]$-modules.*

For the later use, we have to construct a connection between $(\varphi, \hat{G})$-modules and filtered (φ, N)-modules. Let V be a semi-stable representation of G with Hodge-Tate weights in $\{0, \dots, r\}$, $T \subset V$ a G-stable $\mathbb{Z}_p$-lattice and $\hat{\mathfrak{M}} := (\mathfrak{M}, \varphi, \hat{G})$ the $(\varphi, \hat{G})$-module associated to T via Theorem 2.2.1 (1). Let $\mathcal{D} := S_{K_0} \otimes_{\mathfrak{S},\varphi} \mathfrak{M}$. One can extend Frobenius to $\mathcal{D}$ via $\varphi_{\mathcal{D}} = \varphi_{S_{K_0}} \otimes \varphi_{\mathfrak{M}}$. Set $D := \mathcal{D}/(I_+S_{K_0})\mathcal{D}$. Note that $I_+S_{K_0} = \{x \in S_{K_0} | x = \sum_{i=1}^{\infty} a_i u^i, a_i \in K_0\}$. Then D is a finite free K_0-vector space with Frobenius induced from $\mathcal{D}$. Proposition 6.2.1.1 in [Bre97] showed that there exists a unique φ-equivariant section $s : D \hookrightarrow \mathcal{D}$. Therefore $\mathcal{D} = S_{K_0} \otimes_{K_0} D$ by identifying D with $s(D)$.

Now tensoring B^+_{cris} on the both sides of (2.2.2), Note that

$$B^+_{\mathrm{cris}} \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \simeq B^+_{\mathrm{cris}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \simeq B^+_{\mathrm{cris}} \otimes_S \mathcal{D} \simeq B^+_{\mathrm{cris}} \otimes_{K_0} D$$

and set $\hat{\iota}_B = B^+_{\mathrm{cris}} \otimes_{W(R)} \hat{\iota}$, we get the following commutative diagram:

$$
\begin{array}{ccc}
B^+_{\mathrm{cris}} \otimes_{K_0} D & \xrightarrow{\hat{\iota}_B} & \hat{T}^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} B^+_{\mathrm{cris}} \\
\uparrow & & \uparrow \\
W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} & \xrightarrow{\hat{\iota}} & \hat{T}^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R)
\end{array} \tag{2.2.3}
$$

Note there is a G-action on the left side of $\hat{\iota}_B$ induced from that on $W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}$.

On the other hand, the functor D_{st} induces an injection

$$\iota : B^+_{\mathrm{st}} \otimes_{K_0} D_{\mathrm{st}}(V) \longrightarrow V^\vee \otimes_{\mathbb{Q}_p} B^+_{\mathrm{st}}. \tag{2.2.4}$$

such that ι is compatible with φ, N, filtration and G-action on both sides.

Proposition 2.2.2. *Notations as above, there exists a unique K_0-endomorphism $\tilde{N} : D \to D$ such that the G-action on the left side of $\hat{\iota}_B$ is given by*

$$g(a \otimes x) = \sum_{i=0}^{\infty} g(a)\gamma_i(-\log([\underline{\epsilon}(g)])) \otimes \tilde{N}^i(x), \text{ for any } a \in B^+_{\mathrm{cris}},\ x \in D. \tag{2.2.5}$$

Moreover, there exists a K_0-linear isomorphism $i : D_{\mathrm{st}}(V) \to D$ such that i is compatible with Frobenius on both sides and the following diagram commutes:

$$
\begin{array}{ccc}
D_{\mathrm{st}}(V) & \hookrightarrow & T^\vee \otimes_{\mathbb{Z}_p} B^+_{\mathrm{st}} \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{mod}\ \mathfrak{u}} \\
D & \hookrightarrow & T^\vee \otimes_{\mathbb{Z}_p} B^+_{\mathrm{cris}}
\end{array} \tag{2.2.6}
$$

Furthermore, if we identify D with $D_{\mathrm{st}}(V)$ via i, then $N = \tilde{N}$.

Proof. Recall $\mathcal{D} := S_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ with $\varphi_{\mathcal{D}} = \varphi_{S_{K_0}} \otimes \varphi_{\mathfrak{M}}$. One can define filtration $\mathrm{Fil}^i\mathcal{D}$ by $\varphi_{\mathfrak{M}}$ as in [Liu07b], §5.3 (cf. the construction of $\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$). Furthermore, §3.4 and §5.1 in [Liu08] showed there exists a unique monodromy operator $N_{\mathcal{D}}$ on $\mathcal{D}$ such that the data $(\mathcal{D}, \varphi_{\mathcal{D}}, \mathrm{Fil}^i\mathcal{D}, N_{\mathcal{D}})$ is a Breuil module (attached to the filtered (φ, N)-module $D_{\mathrm{st}}(V)$)[5], and the G-action on $B^+_{\mathrm{cris}} \otimes_S \mathcal{D}$, which is the left side of $\hat{\iota}_B$, is given by

$$g(a \otimes x) = \sum_{i=0}^{\infty} g(a)\gamma_i(-\log([\underline{\epsilon}(g)])) \otimes N^i_{\mathcal{D}}(x), \text{ for any } a \in B^+_{\mathrm{cris}},\ x \in \mathcal{D}; \tag{2.2.7}$$

moreover, we have $N_{\mathcal{D}}(D) \subset D$ and for any x in $\mathcal{D}$ and $s \in S$,

$$N_{\mathcal{D}}(sx) = N_S(s)x + sN_{\mathcal{D}}(x). \tag{2.2.8}$$

Now set $\tilde{N} := N_{\mathcal{D}}|_D$. Restricted the G-action to D, we get (2.2.5). To prove the existence of i, we first show that $B^+_{\mathrm{st}} \otimes_{B^+_{\mathrm{cris}}} \hat{\iota}_B = \iota$, i.e., there exists an isomorphism

[5]See [Bre97] or §5.2 in [Liu07b] for axioms of Breuil modules and how to attach Breuil modules to filtered (φ, N)-modules.

$\hat{i}$ of $B^+_{\rm st}$-modules such that the the following diagram commutes

$$(2.2.9)\qquad \begin{array}{ccc} B^+_{\rm st}\otimes_{K_0} D_{\rm st}(V) & \xrightarrow{\ \iota\ } & T^\vee \otimes_{\mathbb{Z}_p} B^+_{\rm st} \\ \hat{i}\uparrow\wr & & \| \\ B^+_{\rm st}\otimes_S \mathcal{D} & \xrightarrow{B^+_{\rm st}\otimes \hat{\iota}_B} & T^\vee \otimes_{\mathbb{Z}_p} B^+_{\rm st} \end{array}$$

where $B^+_{\rm st}\otimes \hat{\iota}_B$ denotes $B^+_{\rm st}\otimes_{B^+_{\rm cris}} \hat{\iota}_B$ (also for the remaining of the proof). Set

$$(2.2.10)\qquad \bar{D} := \left\{ \sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \otimes \tilde{N}^i(y) \in B^+_{\rm st}\otimes_S \mathcal{D} \mid y \in D \right\} \subset B^+_{\rm st}\otimes_S \mathcal{D}.$$

To prove (2.2.9), it suffices to show that $\bar{D} = D_{\rm st}(V)$. Note that $\dim_{K_0}(\bar{D}) = \dim_{K_0}(D) = \mathrm{rank}_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{rank}_{\mathbb{Z}_p}(T)$. It suffices show that $\bar{D}\subset (T^\vee\otimes_{\mathbb{Z}_p} B^+_{\rm st})^G$. But this has been proved in §7.2 in [Liu07b]. This proves the diagram (2.2.9). Furthermore, we have the following commutative diagram

$$(2.2.11)\qquad \begin{array}{ccccc} D_{\rm st}(V) & \hookrightarrow & B^+_{\rm st}\otimes_{K_0} D_{\rm st}(V) & \xrightarrow{\ \iota\ } & T^\vee \otimes_{\mathbb{Z}_p} B^+_{\rm st} \\ \hat{i}|_{\bar{D}}\uparrow\wr & & \hat{i}\uparrow\wr & & \| \\ \bar{D} & \hookrightarrow & B^+_{\rm st}\otimes_S \mathcal{D} & \xrightarrow{B^+_{\rm st}\otimes \hat{\iota}_B} & T^\vee \otimes_{\mathbb{Z}_p} B^+_{\rm st} \end{array}$$

Now modulo $\mathfrak{u}$ on the above diagram and noting that $\bar{D} \mod \mathfrak{u} = D$, we get the commutative diagram (2.2.6). Note that both rows and the right column in diagram (2.2.6) are compatible with Frobenius. So i is compatible with Frobenius. To check that $N = \tilde{N}$ via i, for any $x = \sum\limits_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \otimes \tilde{N}^i(y) \in \bar{D}$ with $y \in D$, we have

$$N(x) = \sum_{i=0}^{\infty} N(\gamma_i(\mathfrak{u})) \otimes \tilde{N}^i(y) = \sum_{i=1}^{\infty} \gamma_{i-1}(\mathfrak{u}) \otimes \tilde{N}^i(y).$$

Hence $i(N(x)) = \tilde{N}(y) = \tilde{N}(i(x))$. Therefore $N = \tilde{N}$. □

Corollary 2.2.3. *The isomorphism i in diagram* (2.2.6) *is unique, and we have $B^+_{\rm st}\otimes_{B^+_{\rm cris}} \hat{\iota}_B = \iota$, i.e., there exists an isomorphism $\hat{i}$ of $B^+_{\rm st}$-modules such that diagram* (2.2.9) *commutes.*

The consequence of the above corollary is that the isomorphism i is functorial. More precisely, let $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Q}_p}$ denote the category of semi-stable representations of G with Hodge-Tate weights in $\{0,\dots,r\}$. For any $V \in \mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Q}_p}$, with $T\subset V$ a G-stable $\mathbb{Z}_p$-lattice and $(\mathfrak{M},\varphi,\hat{G})$ the $(\varphi,\hat{G})$-module attached to T, consider the functor $D(V): V\mapsto \mathcal{D} := S_{K_0}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M} \mapsto D := \mathcal{D}/(I_+S_{K_0})\mathcal{D}$ from $\mathrm{Rep}^{\mathrm{st},r}_{\mathbb{Q}_p}$ to the category of φ-modules. Let $\tilde{D}_{\rm st}(V)$ be the functor $D_{\rm st}(V): V\mapsto D_{\rm st}(V)$ composite by the forgetful functor from the category of filtered (φ,N)-modules to the category of φ-modules.

Corollary 2.2.4. *The isomorphism i induces an isomorphism of functors between $D(\cdot)$ and $\tilde{D}_{\rm st}(\cdot)$.*

2.3. **Construction of M_{st}.** Now let $V \in \mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Q}_p}$ be a p-adic Galois representation of G_K and $T \subset V$ a G_K-stable $\mathbb{Z}_p$-lattice. Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ be the $(\varphi, \hat{G})$-module associated to $T|_G$, $\mathcal{D} := S_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and $D := \mathcal{D}/(I_+S_{K_0})\mathcal{D}$. Put $M := \mathfrak{M}/u\mathfrak{M} \simeq \varphi^*\mathfrak{M}/u\varphi^*\mathfrak{M} \subset \mathcal{D}/(I_+S_{K_0})\mathcal{D} = D$, where $\varphi^*\mathfrak{M} = \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Then M is a finite free $W(k)$-lattice in D and φ-stable. By Corollary 2.2.3, we can fix the unique isomorphism $i : D_{\mathrm{st}}(V) \simeq D$ from now on. Now set $M_{\mathrm{st}}(T) := i^{-1}(M) \subset D_{\mathrm{st}}(V)$. Before we prove that M_{st} satisfies the requirements of Theorem 2.1.3, let us first summarize the construction of M_{st} in the following diagram:

$$\begin{array}{ccccc}
T & \xrightarrow{\text{Theorem 2.2.1}} & \varphi^*\mathfrak{M} & \xrightarrow{\text{mod } u} & M \\
\downarrow & & \downarrow & & \downarrow \\
V & \xrightarrow{S_{K_0}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}} & \mathcal{D} & \xrightarrow{\text{mod } I_+S_{K_0}} & D \\
 & \searrow^{D_{\mathrm{st}}} & & & \uparrow{\scriptstyle i\,\wr} \\
 & & & & D_{\mathrm{st}}(V)
\end{array}
\qquad M \xrightarrow{M_{\mathrm{st}}} D_{\mathrm{st}}(V)$$

Note that Corollary 2.2.4 ensures that the diagram in the bottom triangle is commutative and thus M_{st} is functorial. We postpone the discussion of faithfulness, fullness and exactness of M_{st}, and the case $er < p - 1$ to §2.5.

Now to prove Theorem 2.1.3, we need to show that $M_{\mathrm{st}}(T)$ is a lattice in $D_{\mathrm{st}}(V)$ in the sense of Definition 2.1.2. We have seen that $M_{\mathrm{st}}(T)$ is φ-stable. It remains to prove that $M_{\mathrm{st}}(T)$ is G_K-stable and N-stable. We first need some preparations.

Since $V|_G$ is semi-stable, by Theorem 2.2.1 (2), we have an injection

$$\hat{\iota} : \ W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \longrightarrow \hat{T}^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R). \tag{2.3.1}$$

compatible with G-action on the both sides. Note that T is G_K-stable. So there is a natural G_K-action on the right side of (2.3.1).

Lemma 2.3.1. *The left side of* (2.3.1) *is G_K-stable.*

Proof. Use diagram (2.2.3)

$$\begin{array}{ccc}
B^+_{\mathrm{cris}} \otimes_{K_0} D & \xrightarrow{\hat{\iota}_B} & \hat{T}^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} B^+_{\mathrm{cris}} \\
\uparrow & & \uparrow \\
W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} & \xrightarrow{\hat{\iota}} & \hat{T}^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R)
\end{array}$$

where the right column has the natural G_K-action. We claim that $B^+_{\mathrm{cris}} \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} = B^+_{\mathrm{cris}} \otimes_{K_0} D$ is stable under the G_K-action via $\hat{\iota}_B$. Let us accept the claim and postpone the proof to the end. Select a basis $e_1, \dots, e_d$ of $\hat{\mathfrak{M}}$. For any $g \in G_K$, $g(e_1, \dots, e_d) = (e_1, \dots, e_d)A$ with A a $d \times d$-matrix. It suffices to show that each coefficient of A, say x, is in $W(R)$. To see this, note that $B^+_{\mathrm{cris}} \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}$ is G_K-stable, so $x \in B^+_{\mathrm{cris}}$. On the other hand, by Theorem 2.2.1 (2), $(\varphi(\mathfrak{t}))^r(\hat{T}^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R)) \subset \hat{\iota}(W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}})$. Thus we have $(\varphi(\mathfrak{t}))^r x \in W(R)$. Then $x \in W(R)$ follows Lemma 3.2.2 in [Liu09b].

To prove the claim, it suffices to show that $G_K(D) \subset B^+_{\mathrm{cris}} \otimes_{K_0} D$. By the proof of Proposition 2.2.2, we see that $D_{\mathrm{st}}(V)$ is just $\bar{D} \subset B^+_{\mathrm{st}} \otimes_{K_0} D$ defined in (2.2.10).

Let $\bar{e}_1, \dots, \bar{e}_d$ be a basis of $\bar{D}$ given by $(\bar{e}_1, \dots, \bar{e}_d) = (e_1, \dots, e_d) \sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \bar{N}^i$ with $(e_1, \dots, e_d)$ a basis of D and $\bar{N}$ the matrix of N under the basis $e_1, \dots, e_d$. For any $g \in G_K$, $g(\mathfrak{u}) = \log(g([\underline{\pi}])) = \mathfrak{u} + \lambda$ with $\lambda = \log(\frac{g([\underline{\pi}])}{[\underline{\pi}]})$ in B^+_{cris}. Note that g and N commutes on $\bar{D}$. So if let A_g be the matrix such that $g(\bar{e}_1, \dots, \bar{e}_d) = (\bar{e}_1, \dots, \bar{e}_d)A_g$, we have $A_g \bar{N} = \bar{N} A_g$. Now

$$\begin{aligned} g(e_1, \dots, e_d) &= g(\bar{e}_1, \dots, \bar{e}_d) \sum_{i=0}^{\infty} \gamma_i(-g(\mathfrak{u}))\bar{N}^i \\ &= (\bar{e}_1, \dots, \bar{e}_d) A_g \sum_{i=0}^{\infty} \gamma_i(-\mathfrak{u} - \lambda)\bar{N}^i \\ &= (\bar{e}_1, \dots, \bar{e}_d) \sum_{i=0}^{\infty} \gamma_i(-\mathfrak{u} - \lambda)\bar{N}^i A_g \\ &= (\bar{e}_1, \dots, \bar{e}_d)(\sum_{i=0}^{\infty} \gamma_i(-\mathfrak{u})\bar{N}^i)(\sum_{i=0}^{\infty} \gamma_i(-\lambda)\bar{N}^i A_g) \\ &= (e_1, \dots, e_d)(\sum_{i=0}^{\infty} \gamma_i(-\lambda)\bar{N}^i A_g). \end{aligned}$$

$(\sum_{i=0}^{\infty} \gamma_i(-\lambda)\bar{N}^i A_g)$ converges in B^+_{cris} because N is nilpotent. This proves the claim.

□

Recall that the projection of R to $\bar{k}$ (by modulo I_+R) induces a projection $\nu : W(R) \to W(\bar{k})$ and ν can be naturally extended to $\nu : B^+_{\mathrm{cris}} \to W(\bar{k})[\frac{1}{p}]$. Write $\tilde{K}_0 := W(\bar{k})[\frac{1}{p}]$. We extend ν to $\nu : B^+_{\mathrm{st}} \to \tilde{K}_0$ by sending $\mathfrak{u} \mapsto 0$. Let $I_+B^+_{\mathrm{st}} := \mathrm{Ker}(\nu)$. For any subring $A \subset B^+_{\mathrm{st}}$, put $I_+A = A \cap I_+B^+_{\mathrm{st}}$. For any subring A of B^+_{cris}, this definition of I_+A obviously coincides with the previous one. Following [Fon94a] §5.3, we define

$$I^{[m]}B^+_{\mathrm{cris}} = \{x \in B^+_{\mathrm{cris}} | \varphi^n(x) \in \mathrm{Fil}^m B^+_{\mathrm{cris}}, \text{ for all } n > 0\}$$

and for any subring $A \subset B^+_{\mathrm{cris}}$, write $I^{[m]}A := A \cap I^{[m]}B^+_{\mathrm{cris}}$. By the proof of Lemma 3.2.2 in [Liu09b], $\varphi(\mathfrak{t})$ is a generator of $I^{[1]}W(R)$.

Lemma 2.3.2. (1) $I^{[1]}W(R) \subset I_+W(R)$ *and* $I^{[1]}A_{\mathrm{cris}} \subset I_+A_{\mathrm{cris}}$.
(2) $\nu|_{W(\bar{k})} = \mathrm{Id}$ *and* ν *is* G_K*-equivariant.*

Proof. (1) It suffices to show that $I^{[1]}A_{\mathrm{cris}} \subset I_+A_{\mathrm{cris}}$. By Proposition 5.3.1 in [Fon94a], for any $x \in I^{[1]}A_{\mathrm{cris}}$, $x = \sum_{n=1}^{\infty} a_i t^{\{n\}}$ where $t = \log([\underline{\epsilon}])$ and $\underline{\epsilon} = (\epsilon_i)_{i \geq 0} \in R$ with ϵ_i primitive p^i-th root of unity. Since $\nu([\underline{\epsilon}]) = 1$, we see that $t \in I_+A_{\mathrm{cris}}$ and then $x \in I_+A_{\mathrm{cris}}$.

(2) The first statement is obvious. To show the second, note that the projection $\bar{\nu} : R \to \bar{k}$ is natural. So ν is obviously G_K-equivariant on $W(R)$, thus on A_{cris} and B^+_{cris}. Therefore it suffices to show that $G_K(\mathfrak{u}) \subset I_+B^+_{\mathrm{st}}$. For any $g \in G_K$, as in the proof of Lemma 2.3.1, $g(\mathfrak{u}) = \log(g([\underline{\pi}])) = \mathfrak{u} + \lambda$ with $\lambda = \log(\frac{g([\underline{\pi}])}{[\underline{\pi}]})$ in B^+_{cris}. It suffices to show that λ is in $I_+B^+_{\mathrm{cris}}$. Write $\underline{\eta} = \frac{g(\underline{\pi})}{\underline{\pi}} = (\eta_i)_{i \geq 0} \in R$. Since

$m\lambda = m\log([\underline{\eta}]) = \log([\underline{\eta}^m])$, after replacing $\underline{\eta}$ by some power, we can assume that $\eta_0 = 1 \mod p$. Hence $\nu([\underline{\eta}]) = 1$ and $\lambda = \log([\underline{\eta}])$ is in $I_+B^+_{\mathrm{cris}}$. □

Now recall that $M = \mathfrak{M}/u\mathfrak{M}$ and $D = \mathcal{D}/(I_+S_{K_0})\mathcal{D}$ in the construction of M_{st}. We have

$$W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \mod I_+W(R) \simeq W(\bar{k}) \otimes_{W(k)} M$$

and

$$B^+_{\mathrm{st}} \otimes_S \mathcal{D} \mod I_+B^+_{\mathrm{st}} \simeq \tilde{K}_0 \otimes_{K_0} D.$$

By Lemma 2.3.1 and Lemma 2.3.2, we have $W(\bar{k})$-semi-linear G_K-actions on $W(\bar{k}) \otimes_{W(k)} M$ and $\tilde{K}_0 \otimes_{K_0} D$. By Proposition 2.2.2, the isomorphism i induces a $W(k)$-linear isomorphism

$$B^+_{\mathrm{st}} \otimes_S \mathcal{D} \mod I_+B^+_{\mathrm{st}} \simeq \tilde{K}_0 \otimes_{K_0} D \xrightarrow[\sim]{\tilde{K}_0 \otimes_{K_0} i^{-1}} \tilde{K}_0 \otimes_{K_0} D_{\mathrm{st}}(V). \tag{2.3.2}$$

On the other hand, note that the right side also has a natural G_K-action induced from the G_K-action on $D_{\mathrm{st}}(V)$.

Lemma 2.3.3. *$\tilde{K}_0 \otimes_{K_0} i^{-1}$ is G_K-equivariant.*

Proof. Combining Corollary 2.2.3 with Lemma 2.3.1, the isomorphism

$$\hat{i}: \ B^+_{\mathrm{st}} \otimes_S \mathcal{D} \xrightarrow{\sim} B^+_{\mathrm{st}} \otimes_{K_0} D_{\mathrm{st}}(V)$$

is G_K-equivariant (note that Corollary 2.2.3 only shows that $\hat{i}$ is G-equivariant). Now modulo $I_+B^+_{\mathrm{st}}$ on the both sides of $\hat{i}$, we see that $\tilde{K}_0 \otimes_{K_0} i^{-1}$ is G_K-equivariant. □

Now since the G_K-action on $W(\bar{k}) \otimes_{W(k)} M$ is stable in $\tilde{K}_0 \otimes_{K_0} D$, we prove:

Corollary 2.3.4. *$M_{\mathrm{st}}(T)$ is G_K-stable in $D_{\mathrm{st}}(V)$.*

2.4. **Stability of monodromy.** Now let us prove that $M_{\mathrm{st}}(T)$ is stable under monodromy. For this, we need more precise information on the Galois action on the left side of (2.3.1). It will be always convenient to consider $W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}$ as a submodule of $B^+_{\mathrm{cris}} \otimes_{K_0} D \simeq B^+_{\mathrm{cris}} \otimes_S \mathcal{D}$, as illustrated in Diagram (2.2.3). In the proof of Proposition 2.2.2, we have shown that there exists an operator $N_{\mathcal{D}}: \mathcal{D} \to \mathcal{D}$ such that the G-action on $B^+_{\mathrm{cris}} \otimes_S \mathcal{D}$ is given by (2.2.7), and formula (2.2.8) show the following diagram is commutative

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{N_{\mathcal{D}}} & \mathcal{D} \\ {\scriptstyle \mod (I_+S_{K_0})\mathcal{D}} \downarrow & & \downarrow {\scriptstyle \mod (I_+S_{K_0})\mathcal{D}} \\ D & \xrightarrow{N} & D \end{array} \tag{2.4.1}$$

Now set $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. Note that $\mathcal{M}/(I_+S)\mathcal{M} \simeq \mathfrak{M}/u\mathfrak{M}$. Hence to show that M_{st} is stable under N, it suffices to show the following:

Proposition 2.4.1. *Notations as above. If $p > 2$ then $N_{\mathcal{D}}(\mathcal{M}) \subset \mathcal{M}$.*

Remark 2.4.2. This generalizes Lemma 3.5.3 in [Liu08] to case that has no restriction on Hodge-Tate weights.

Proof of Proposition 2.4.1. Regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $\mathcal{M}$. It suffices to show that $N_{\mathcal{D}}(\mathfrak{M}) \subset \mathcal{M}$. Since $p > 2$, by Lemma 5.1.2 in [Liu08], we have $K'_{p^\infty} \cap K'_\infty = K'$, $\hat{G} = G_{p^\infty} \rtimes H_{K'}$ and $G_{p^\infty} \simeq \mathbb{Z}_p(1)$. As the consequence, let τ be a topological generator of G_{p^∞} then $\underline{\epsilon}(\tau) = (\epsilon_i)_{i \geq 0} \in R$ with ϵ_i a p^i-th primitive root of unity. So we may just select $\underline{\epsilon} = \underline{\epsilon}(\tau)$ and set $t = -\log([\underline{\epsilon}(\tau)])$. Pick a $x \in \mathfrak{M}$. By formula (2.2.7), we have $\tau(x) = \sum\limits_{i=0}^{\infty} \gamma_i(t) \otimes N^i_{\mathcal{D}}(x)$. Note that τ acts on t trivially. Hence for any $n \geq 1$ and $x \in \mathcal{D}$, an easy induction on n shows that

$$(\tau - 1)^n(x) = \sum_{m=n}^{\infty} \left(\sum_{i_1 + \cdots + i_n = m, i_j \geq 1} \frac{m!}{i_1! \cdots i_n!} \right) \gamma_m(t) \otimes N^m_{\mathcal{D}}(x) \tag{2.4.2}$$

In particular, $(\tau - 1)^n(x) \in I^{[n]} B^+_{\mathrm{cris}} \otimes_S \mathcal{D}$. Since $x \in \mathfrak{M}$ and $W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ is G-stable via $\hat{\iota}$. We get $(\tau - 1)^n(x) \in I^{[n]} W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$.

Now we claim that $\frac{(\tau-1)^n}{nt}(x)$ is well defined in $A_{\mathrm{cris}} \otimes_S \mathcal{M}$ and $\frac{(\tau-1)^n}{nt}(x) \to 0$ p-adically as $n \to \infty$. If so, then we can define

$$\frac{\log}{t}(\tau)(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\tau - 1)^n}{nt}(x) \in A_{\mathrm{cris}} \otimes_S \mathcal{M}. \tag{2.4.3}$$

and a direct computation shows that

$$\frac{\log}{t}(\tau)(x) = 1 \otimes N_{\mathcal{D}}(x) \in A_{\mathrm{cris}} \otimes_S \mathcal{M}. \tag{2.4.4}$$

Therefore, we see that $N_{\mathcal{D}}(x) \in \mathcal{M}$ and we are done. It suffices to show the claim. Note that $t = c\varphi(\mathfrak{t})$ with $\varphi(\mathfrak{t})$ a generator of $I^{[1]}W(R)$, and c a unit in S. Since $(\tau - 1)^n(x) \in I^{[n]} W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$, it suffices to show that for any n, $\frac{(\varphi(\mathfrak{t}))^{n-1}}{n}$ is in A_{cris} and it goes to zero p-adically. Note that $(\varphi(\mathfrak{t}))^{p-1} \in \mathrm{Fil}^p W(R) + pW(R)$. So $\frac{(\varphi(\mathfrak{t}))^{p-1}}{p}$ is in A_{cris}. For any $n \geq 2p$, write $n = p^s m$ with $p \nmid m$. We see that

$$\frac{(\varphi(\mathfrak{t}))^{n-1}}{n} = \frac{(\varphi(\mathfrak{t}))^{n-1}}{(n-1)!} \frac{(n-1)!}{n} = \frac{(\varphi(\mathfrak{t}))^{n-1}}{(n-1)!} \frac{(p^s m - 1)!}{p^s m}$$

Since $\gamma_i(t)$ goes to zero p-adically (see [Fon94a] §5.2.4), it suffices to show $\frac{(p^s m-1)!}{p^s m}$ is in $\mathbb{Z}_p$. If $m > 1$ then $p^s m - 1 > p^s$ and it is obvious. If $m = 1$ then one easily check that $p^s - 1 \geq 2p^{s-1}$ because $s \geq 2$ and $p \geq 3$. $v_p((p^s - 1)!) \geq v_p(2p^{s-1}) + v_p(p^{s-1}) = 2(s-1) \geq s$.

□

To complete the proof that $N(M_{\mathrm{st}}(T)) \subset M_{\mathrm{st}}(T)$ when $p = 2$ and to deal with torsion representations later, it will be convenient to give another description of monodromy N on $M_{\mathrm{st}}(T)$. As the proof of the above proposition, if $p > 2$ then we select τ a topological generator of G_{p^∞} and set $t = -\log([\underline{\epsilon}(\tau)])$. If $p = 2$ then there still exists a $\tau \in \hat{G}$ such that $\underline{\epsilon}(\tau) = (\eta_i)_{i \geq 0} \in R$ with η_i being *primitive* p^i-th root of unity (see the end of §4.1 in [Liu09b]). We set $t := -\log([\underline{\epsilon}(\tau)])$ in this case. Recall that $\varphi^*\mathfrak{M} := \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$, which is a submodule of $\mathcal{D}$ and $W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. For any $x \in \varphi^*\mathfrak{M}$, by the proof of the above theorem, we see that $(\tau - 1)(x) \in I^{[1]} W(R) \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M}$. Note that the statement is also valid for $p = 2$

because the same proof works for our selection of τ when $p=2$. Since $\varphi(\mathfrak{t})$ is a generator of $I^{[1]}W(R)$, it makes sense to define a map

$$\hat{N}:\varphi^*\mathfrak{M}\to W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M};\quad \hat{N}(x)=\frac{\tau-1}{\varphi(\mathfrak{t})}(x). \tag{2.4.5}$$

It is easy to check that $\hat{N}$ is $W(k)$-linear and $\hat{N}((I_+\mathfrak{S})\varphi^*\mathfrak{M})\subset I_+W(R)\otimes_{\mathfrak{S}}\varphi^*\mathfrak{M}$. So it make senses to define a map $\tilde{N}=\hat{N}(x) \mod I_+W(R)$, that is, we have the following commutative diagram:

$$\begin{array}{ccc} \varphi^*\mathfrak{M} & \xrightarrow{\hat{N}} & W(R)\otimes_{\mathfrak{S}}\varphi^*\mathfrak{M} \\ {\scriptstyle \mod I_+\mathfrak{S}}\downarrow & & \downarrow{\scriptstyle \mod I_+W(R)} \\ M_{\mathrm{st}}(T) & \xrightarrow{\tilde{N}} & W(\bar{k})\otimes_{W(k)} M_{\mathrm{st}}(T) \end{array} \tag{2.4.6}$$

On the other hand, the monodromy operator N define a map $N: D\to D\hookrightarrow D\otimes_{K_0}\tilde{K}_0$.

Proposition 2.4.3. *Notations as above, $\tilde{N}=N|_{M_{\mathrm{st}}(T)}$ and then $N(M_{\mathrm{st}}(T))\subset M_{\mathrm{st}}(T)$.*

Proof. For any $x\in\varphi^*\mathfrak{M}\subset\mathcal{M}=S\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$, we have $(\tau-1)(x)=\sum\limits_{i=1}^{\infty}\gamma_i(t)\otimes N^i_{\mathcal{D}}(x)$. Recall that $t=c\varphi(\mathfrak{t})$. Hence $\hat{N}(x)=cN_{\mathcal{D}}(x)+\sum\limits_{i=2}^{\infty}\frac{ct^{i-1}}{i!}\otimes N^i_{\mathcal{D}}(x)$. Now we claim that the series $\sum\limits_{i=2}^{\infty}\frac{ct^{i-1}}{i!}\otimes N^i_{\mathcal{D}}(x)$ converges in $B^+_{\mathrm{cris}}\otimes_{\mathfrak{S}}\varphi^*\mathfrak{M}$. Let us first accept this and postpone the proof to the end. Now since $t\in I_+A_{\mathrm{cris}}$, we see that $\hat{N}(x)-cN_{\mathcal{D}}(x)\in I_+B^+_{\mathrm{cris}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$. But $c=1 \mod I_+S$, we have $\hat{N}(x)=N_{\mathcal{D}}(x) \mod I_+B^+_{\mathrm{cris}}\otimes_{\mathfrak{S}}\varphi^*\mathfrak{M}$. Since $N_{\mathcal{D}}=N \mod I_+S_{K_0}$, we see that $\tilde{N}=N|_{M_{\mathrm{st}}(T)}$.

To prove the claim, we need check two facts: First, $\frac{t^i}{(i+1)!}\to 0$ p-adically in B^+_{cris} when i goes to infinity; second, there exist an integer m such that $p^mN^i_{\mathcal{D}}(x)\in\mathcal{M}=S\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$ for all i. To see the first fact, recall that $t^{\{i\}}=\frac{t^i}{p^{\tilde{q}(i)}\tilde{q}(i)!}\in A_{\mathrm{cris}}$, where $i=(p-1)\tilde{q}(i)+r$ with $0\le r<p-1$. It suffices to show that $p^{\tilde{q}(i)}\tilde{q}(i)!/(i+1)!$ goes to zero p-adically. This easily follows form the well-known fact that $v_p(m!)<\frac{m}{p-1}$. To prove the second fact, note that $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathcal{M}\simeq S\otimes_{W(k)}D$. Hence $x=\sum_j s_j\otimes y_i$ with $s_i\in S$ and $y_i\in D$. Then second fact follows the facts that $N_{\mathcal{D}}=N_S\otimes 1+1\otimes N_D$ (cf. (2.2.8)) and that N_D on D is nilpotent.

□

2.5. **Some properties of M_{st}.** In this subsection, we discuss the properties of the functor M_{st} stated in Theorem 2.1.3.

To show that M_{st} is faithful, let $f:T\to T'$ be a map in $\mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Z}_p}$ such that $M_{\mathrm{st}}(f)=0$. Then $D_{\mathrm{st}}(f)=M_{\mathrm{st}}(f)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p=0$ and then $f\otimes_{\mathbb{Z}_p}\mathbb{Q}_p=0$. Therefore $f=0$ and the faithfulness of M_{st} is proved.

Now let us discuss the fullness of M_{st}. Let T, $T'\in\mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Z}_p}$ be two G_K-stable $\mathbb{Z}_p$-lattices and $f: M_{\mathrm{st}}(T')\to M_{\mathrm{st}}(T)$ a morphism in $\mathrm{L}^r(\varphi,N,\Gamma)$. Tensoring $\mathbb{Q}_p$, $f\otimes\mathbb{Q}_p$ is a morphism of filtered (φ,N,Γ)-modules. Hence there exists a morphism

$g : T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to T' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ inside $\mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Q}_p}$ such that $D_{\mathrm{st}}(g) = f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. To show that M_{st} is full, it suffices to show that $g(T) \subset T'$. Pick a constant a such that $p^a g(T) \subset T'$ and let $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G}_{\mathfrak{M}})$ and $(\mathfrak{M}', \varphi_{\mathfrak{M}'}, \hat{G}_{\mathfrak{M}'})$ be the $(\varphi, \hat{G})$-modules corresponding to T and T' respectively. Then $p^a g$ induces a morphism $\mathfrak{f} : (\mathfrak{M}', \varphi_{\mathfrak{M}'}, \hat{G}_{\mathfrak{M}'}) \to (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G}_{\mathfrak{M}})$ of $(\varphi, \hat{G})$-modules. To show that M_{st} is full, it suffices to show that $\mathfrak{f}(\mathfrak{M}') \subset p^a\mathfrak{M}$. Write $\bar{\mathfrak{f}} = \mathfrak{f} \mod u : \mathfrak{M}'/u\mathfrak{M}' \to \mathfrak{M}/u\mathfrak{M}$. By the construction of M_{st}, We see that $\bar{\mathfrak{f}} = M_{\mathrm{st}}(p^a g) = p^a f$.

Proposition 2.5.1. *Suppose $er < p-1$ and $\mathfrak{f} : \mathfrak{M}' \to \mathfrak{M}$ is a morphism of finite free Kisin modules of height r. Write $M' := \mathfrak{M}' \mod u\mathfrak{S}$, $M := \mathfrak{M} \mod u\mathfrak{S}$ and $\bar{\mathfrak{f}} = \mathfrak{f} \mod u\mathfrak{S}$. If $\bar{\mathfrak{f}}(M') \subset p^a M$ then $\mathfrak{f}(\mathfrak{M}') \subset p^a\mathfrak{M}$.*

Corollary 2.5.2. *If $er < p-1$. M_{st} is fully faithful.*

To prove this proposition, we need the following result:

Lemma 2.5.3. *Assume that $er < p-1$.*

(1) *Suppose that $\mathfrak{i} : \mathfrak{L} \hookrightarrow \mathfrak{L}'$ is an injective morphism inside $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ and $\mathfrak{M} := \mathfrak{L}/\mathfrak{i}(\mathfrak{L}')$ is killed by some p-power. Then as an $\mathfrak{S}$-module, $\mathfrak{M} \simeq \mathfrak{S}/p^{n_1}\mathfrak{S} \oplus \mathfrak{S}/p^{n_2}\mathfrak{S} \oplus \cdots \oplus \mathfrak{S}/p^{n_d}\mathfrak{S}$ with $n_i \geq 1$.*
(2) *Suppose $\mathfrak{M}$ is a torsion free φ-module of height r. Then $\mathfrak{M}$ is finite $\mathfrak{S}$-free.*

The proof of the lemma needs an elaborate discussion of torsion Kisin modules. So we postpone the proof to §3.2.

Proof of Proposition 2.5.1. Let $\mathfrak{L}' = \mathfrak{M}/\mathfrak{f}(\mathfrak{M}')$, $\mathfrak{L}'_{\mathrm{tor}}$ the torsion part of $\mathfrak{L}'$ and $\mathfrak{L} = \mathfrak{L}'/\mathfrak{L}'_{\mathrm{tor}}$. So we get an exact sequence of φ-modules:

$$0 \to \mathfrak{N} \to \mathfrak{M} \to \mathfrak{L} \to 0$$

with $\mathfrak{N}$ and $\mathfrak{L}$ torsion free. By Proposition B 1.3.5 in [Fon90], we see that $\mathfrak{N}$, $\mathfrak{L}$ are of height r. Then Lemma 2.5.3 shows that $\mathfrak{N}$ and $\mathfrak{L}$ are finite free. Similarly, we have an exact sequence of finite free Kisin modules $0 \to \mathfrak{K} \to \mathfrak{M}' \xrightarrow{\mathfrak{f}} \mathfrak{f}(\mathfrak{M}') \to 0$. Write $\mathfrak{N}' = \mathfrak{f}(\mathfrak{M}')$. We have an injection $\mathfrak{N}' \hookrightarrow \mathfrak{N}$. Since $\mathfrak{N}/\mathfrak{N}' \simeq \mathfrak{L}'_{\mathrm{tor}}$, by Lemma 2.5.3 (1), we have

$$\mathfrak{N}/\mathfrak{N}' \simeq \mathfrak{S}/p^{n_1}\mathfrak{S} \oplus \mathfrak{S}/p^{n_2}\mathfrak{S} \oplus \cdots \oplus \mathfrak{S}/p^{n_d}\mathfrak{S}. \quad (2.5.1)$$

In particular, we get an exact sequence $0 \to \mathfrak{N}' \to \mathfrak{N} \to \mathfrak{N}/\mathfrak{N}' \to 0$ with $\mathfrak{N}/\mathfrak{N}'$ u-torsion free. Since all terms of three exact sequences here are u-torsion free, after modulo u, we still get exact sequences. Hence the map $\bar{\mathfrak{f}} : M' \to M$ can be decomposed into three parts: $M' \twoheadrightarrow N'$, $N' \hookrightarrow N$ and $N \hookrightarrow M$, where N and N' are $\mathfrak{N} \mod u$ and $\mathfrak{N}' \mod u$ respectively. Note $\bar{\mathfrak{f}}(M') = N' \subset N \cap p^a M$. Since $M/N = \mathfrak{L} \mod u$ is torsion free, we see that $N \cap p^a M = p^a N$ and thus $N' \subset p^a N$. Write $\mathfrak{g} : \mathfrak{N}' \hookrightarrow \mathfrak{N}$ and $g = \mathfrak{g} \mod u : N' \to N$. Now it suffices to show Proposition 2.5.1 for $\mathfrak{g} : \mathfrak{N}' \hookrightarrow \mathfrak{N}$. To see this, it is enough to show that n_i in (2.5.1) satisfies $n_i \geq a$. But $N' \subset p^a N$ and

$$N/N' \simeq (\mathfrak{N}/\mathfrak{N}') \mod u = W(k)/p^{n_1}W(k) \oplus W(k)/p^{n_2}W(k) \oplus \cdots \oplus W(k)/p^{n_d}W(k).$$

We must have $n_i \geq a$. □

Let us prove the left exactness, and the exactness of M_{st} if $er < p-1$. Let $0 \to T'' \to T \to T' \to 0$ be an exact sequence in $\mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Z}_p}$. Restricted to G, we get a sequence of $(\varphi, \hat G)$-modules $0 \to \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'' \to 0$. Write

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$$

for the corresponding sequence of Kisin modules. It suffices to show that the above sequence is left exact and exact when $er < p-1$.

Lemma 2.5.4. *Let $\mathfrak{M}'$, $\mathfrak{M}''$ and $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ be finite free Kisin modules. Assume that there exists an exact sequence of $\mathbb{Z}_p[G_\infty]$-modules*

$$0 \to T_{\mathfrak{S}}(\mathfrak{M}'') \to T_{\mathfrak{S}}(\mathfrak{M}) \to T_{\mathfrak{S}}(\mathfrak{M}') \to 0 \tag{2.5.2}$$

Then there exists a (unique) left exact sequence of Kisin modules

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0 \tag{2.5.3}$$

such that $T_{\mathfrak{S}}((2.5.3)) \simeq (2.5.2)$. If $er < p-1$ then (2.5.3) is exact.

Proof. Since $T_{\mathfrak{S}}$ is a fully faithful functor (see, for example, Corollary 4.2.6 in [Liu07b]), we have a short sequence

$$0 \to \mathfrak{M}' \to \mathfrak{M} \xrightarrow{f} \mathfrak{M}'' \to 0 \tag{2.5.4}$$

of finite free Kisin modules such that $T_{\mathfrak{S}}((2.5.4)) \simeq (2.5.2)$. The only question is whether the above sequence is left exact. In the following, we heavily use the theory of *étale* φ-modules, their attached representations of G_∞, and relations to Kisin modules. We refer readers to §2 in [Liu07b] for more details.

For any finite free Kisin module $\mathfrak{N}$, set $N := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N}$. Following [Fon90], then N is a finite $\mathcal{O}_{\mathcal{E}}$-free étale φ-module, in the sense that N has a semi-linear Frobenius $\varphi : N \to N$ and the $\mathcal{O}_{\mathcal{E}}$-linear map $1 \otimes \varphi : \mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} N \to N$ is an bijection. Write $T(N) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi}(N, \mathcal{O}_{\hat{\mathcal{E}}^{\mathrm{ur}}})$. By Corollary 2.2.2 in [Liu07b], $T(N) \simeq T_{\mathfrak{S}}(\mathfrak{N})$ as $\mathbb{Z}_p[G_\infty]$-modules. Moreover, Proposition A 1.2.4, Proposition A 1.2.6 and Remark A 1.2.7 in [Fon90] proved that $N = \mathrm{Hom}_{G_\infty}(T(N), \mathcal{O}_{\hat{\mathcal{E}}^{\mathrm{ur}}})$ and the functor $T \mapsto \mathrm{Hom}_{G_\infty}(T, \mathcal{O}_{\hat{\mathcal{E}}^{\mathrm{ur}}})$ induces an exact anti-equivalence between $\mathrm{Rep}(G_\infty)$ and the category of étale φ-modules. So write $M' = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}'$, $M = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ and $M'' = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}''$. Applying $\mathrm{Hom}_{G_\infty}(-, \mathcal{O}_{\hat{\mathcal{E}}^{\mathrm{ur}}})$ to the exact sequence (2.5.2), we get an exact sequence $0 \to M' \to M \to M'' \to 0$, which is just $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} (2.5.4)$. Hence $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} (2.5.4)$ is exact. Set $\tilde{\mathfrak{M}}'' = f(\mathfrak{M})$ and $\tilde{\mathfrak{M}}' = \mathrm{Ker}(f|_{\mathfrak{M}})$. Apparently, $\tilde{\mathfrak{M}}'$ and $\tilde{\mathfrak{M}}''$ are φ-modules and torsion free. By Proposition B 1.3.5 in [Fon90], we see that $\tilde{\mathfrak{M}}'$ and $\tilde{\mathfrak{M}}''$ are objects in $'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$. Lemma 2.3.7 in [Liu07b] shows that there exist finite free Kisin modules $\overline{\mathfrak{M}}'$ and $\overline{\mathfrak{M}}''$ such that $\tilde{\mathfrak{M}}' \subset \overline{\mathfrak{M}}' \subset M'$ and $\tilde{\mathfrak{M}}'' \subset \overline{\mathfrak{M}}'' \subset M''$. But $T_{\mathfrak{S}}$ is fully faithful and $\mathfrak{M}' \subset \tilde{\mathfrak{M}}' \subset M'$ is also a finite free Kisin module. Noting that $T_{\mathfrak{S}}(\mathfrak{M}') = T_{\mathfrak{S}}(\overline{\mathfrak{M}}') = T(M')$, we must have $\mathfrak{M}' = \tilde{\mathfrak{M}}' = \overline{\mathfrak{M}}'$. Similarly, we have $\overline{\mathfrak{M}}'' = \mathfrak{M}''$. Hence we get a left exact sequence

$$0 \to \mathfrak{M}' \to \mathfrak{M} \twoheadrightarrow \tilde{\mathfrak{M}}'' \hookrightarrow \mathfrak{M}''$$

So (2.5.4) is left exact. In general, $\tilde{\mathfrak{M}}''$ may not be $\mathfrak{M}''$. See Example 2.5.6. But if $er < p-1$. By Lemma 2.5.3 (1), we see that $\tilde{\mathfrak{M}}''$ is finite free. Hence $\tilde{\mathfrak{M}}'' = \mathfrak{M}''$ by the full faithfulness of $T_{\mathfrak{S}}$ and then (2.5.3) is exact.

□

In general. M_{st} is neither full nor exact if $er \geq p-1$. The following examples restrict to crystalline representations with Hodge-Tate weights in $\{0,1\}$. In this situation, Theorem (0.4) in [Kis06] shows that the category of finite free Kisin modules of height 1 is equivalent to the category of G-stable $\mathbb{Z}_p$-lattices in crystalline representations with Hodge-Tate weights in $\{0,1\}$. In particular, for each finite free Kisin module $\mathfrak{M}$ of height 1, there exists a unique (φ, G)-module $\hat{\mathfrak{M}}$ such that $\hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a crystalline representation with Hodge-Tate weights in $\{0,1\}$ and the ambient Kisin module of $\hat{\mathfrak{M}}$ is just $\mathfrak{M}$.

Example 2.5.5. Let $K = \mathbb{Q}_p(\pi)$ with $\pi^{p-1} = p$. So we have $E(u) = u^{p-1} - p$. Let T be a G-stable $\mathbb{Z}_p$-lattice in a 2-dimensional crystalline representation with Hodge-Tate weights $\{0,1\}$, determined by Kisin module $\mathfrak{M} = \mathfrak{S}e_1 \oplus \mathfrak{S}e_2$ and $\varphi(e_1, e_2) = (e_1, e_2)\begin{pmatrix} 1 & 0 \\ 0 & E(u) \end{pmatrix}$. Let $\mathfrak{M}' = \mathfrak{S}f_1 \oplus \mathfrak{S}f_2 \subset \mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ be a finite free $\mathfrak{S}$-module with the basis $(f_1, f_2) = (e_1, e_2)\begin{pmatrix} 1 & \frac{u}{p} \\ 0 & 1 \end{pmatrix}$. Note that

$$\varphi(f_1, f_2) = (f_1, f_2)\begin{pmatrix} 1 & \frac{u}{p} \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & E(u) \end{pmatrix}\begin{pmatrix} 1 & \frac{u^p}{p} \\ 0 & 1 \end{pmatrix} = (f_1, f_2)\begin{pmatrix} 1 & u \\ 0 & E(u) \end{pmatrix}.$$

Hence $\mathfrak{M}'$ is a Kisin module of height 1 and corresponds a G-stable $\mathbb{Z}_p$-lattice T' inside $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. $T \neq T'$ because $\mathfrak{M}' \neq \mathfrak{M}$. But $\mathfrak{M}/u\mathfrak{M} = \mathfrak{M}'/u\mathfrak{M}'$ inside $\mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and thus $M_{\mathrm{st}}(T) = M_{\mathrm{st}}(T')$ inside $D_{\mathrm{st}}(T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$.

Example 2.5.6. Let $E(u)$, K and $\mathfrak{M}'$ be the same as the above example. Write $\mathfrak{S}^*$ for the rank-1 finite free Kisin module whose Frobenius is given by $\varphi(f) = E(u)f$ with f a basis of $\mathfrak{S}^*$, and $\mathfrak{S}$ for the rank-1 finite free Kisin module with trivial Frobnius. We have a left exact sequence of Kisin modules

$$0 \to \mathfrak{S}^* \to \mathfrak{M}' \xrightarrow{\mathfrak{f}} \mathfrak{S} \to 0$$

via $\mathfrak{f}(f_1) = p$ and $\mathfrak{f}(f_2) = u$. The above sequence of Kisin modules corresponds to an exact sequence of lattices in crystalline representations with Hodge-Tate weights in $\{0,1\}$. Modulo u on the above sequence, we see easily that M_{st} is not right exact.

3. p-adic Hodge data of torsion representations

Let T be a p-adic Galois representation killed by some p-power, and $\mathbf{P}$ some properties of representations (e.g., being semi-stable, crystalline etc.). We call a representation is *torsion* $\mathbf{P}$ if T can be written as a quotient of two lattices inside a representation which has property $\mathbf{P}$. In this section, we will attach (φ, N)-module to torsion potentially semi-stable representation up to a p-power $p^{\mathfrak{c}}$, where $\mathfrak{c}$ is a constant only depending on the absolute ramification index e (of the field over which representation is semi-stable) and length of Hodge-Tate weights.

We follow the same settings and notations in §2 throughout this section.

3.1. Construction of M_{st} for torsion representations.

Recall that $\mathrm{Rep}^{\mathrm{pst},r}_{\mathbb{Q}_p}$ is the category of potentially semi-stable representations (satisfying the assumption in the beginning of §2) with Hodge-Tate weights in $\{0,\dots,r\}$ where r is a fixed positive integer. We denote $\mathrm{Rep}^{\mathrm{pst},r}_{\mathrm{tor}}$ the category whose objects are *torsion potentially semi-stable representations with Hodge-Tate weights in* $\{0,\dots,r\}$, in the sense that, for any $T \in \mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},r}$, there exist G_K-stable $\mathbb{Z}_p$-lattices $L \subset L'$ in a $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{pst},r}$ such that $T \simeq L'/L$ as $\mathbb{Z}_p[G_K]$-modules. We call the pair $L \subset L'$ a *lift* of T. Obviously, for any $T \in \mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},r}$, the lift is always not unique. A morphism between two lifts $\mathcal{L} : L \subset L'$ (lifting T) and $\tilde{\mathcal{L}} : \tilde{L} \subset \tilde{L}'$ (lifting $\tilde{T}$) is a morphism $\hat{f} : L' \to \tilde{L}'$ in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{pst},r}$ such that $\hat{f}(L) \subset \tilde{L}$. $\hat{f}$ induces a morphism $f : T \to \tilde{T}$ in $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},r}$. We call $\hat{f}$ a *lift of* f (with respect to lifts $\mathcal{L}$ and $\tilde{\mathcal{L}}$).

Let $\mathcal{L} : L \overset{j}{\hookrightarrow} L'$ be a lift of T. By Theorem 2.1.3, we get a morphism $M_{\mathrm{st}}(j) : M_{\mathrm{st}}(L') \to M_{\mathrm{st}}(L)$ in $\mathrm{L}^r(\varphi, N, \Gamma)$ ($M_{\mathrm{st}}(j)$ is injective by Corollary 3.2.4 below). Now write $\tilde{j} := M_{\mathrm{st}}(j)$ and set $M_{\mathrm{st},\mathcal{L}}(T) := M_{\mathrm{st}}(L)/\tilde{j}(M_{\mathrm{st}}(L'))$. Then $M_{\mathrm{st},\mathcal{L}}(T)$ has G_K-action, Frobenius φ and monodromy N induced from $M_{\mathrm{st}}(L)$. Following the definition of filtered (φ, N, Γ)-modules, we define a category $M_{\mathrm{tor}}(\varphi, N, \Gamma)$ whose objects are finite $W(k)$-modules M killed by some p-power and endowed with

- a Frobenius semi-linear map: $\varphi : D \to D$.
- a $W(k)$-linear map $N : D \to D$ such that $N\varphi = p\varphi N$.
- a $W(k)$-linear Γ-action on D such that Γ commutes with φ and N.

We have obvious notations of morphisms in $M_{\mathrm{tor}}(\varphi, N, \Gamma)$. Therefore $M_{\mathrm{st},\mathcal{L}}(T)$ is an object in $M_{\mathrm{tor}}(\varphi, N, \Gamma)$. If $f : T \to \tilde{T}$ is a morphism in $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},r}$ which can be lifted to $\hat{f} : \mathcal{L} \to \tilde{\mathcal{L}}$, then $\hat{f}$ induces a morphism $M_{\mathrm{st},\hat{f}}(f) : M_{\mathrm{st},\tilde{\mathcal{L}}}(\tilde{T}) \to M_{\mathrm{st},\mathcal{L}}(T)$ in $M_{\mathrm{tor}}(\varphi, N, \Gamma)$.

Though the above definitions depend on the choices of the lifts, we will prove that there exists a constant $\mathfrak{c}$ such that $M_{\mathrm{st},\mathcal{L}}(T)$ is "unique" up to $p^{\mathfrak{c}}$ in the following sense:

Theorem 3.1.1. *There exists a constant $\mathfrak{c}$ only depending on e and r such that the following statement holds: for any morphism $f : T' \to T$ in $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},r}$ and any lift $\mathcal{L}'$, $\mathcal{L}$ of T', T respectively, there exists a morphism $g : M_{\mathrm{st},\mathcal{L}}(T) \to M_{\mathrm{st},\mathcal{L}'}(T')$ in $M_{\mathrm{tor}}(\varphi, N, \Gamma)$ such that*

(1) *if there exists a morphism of lifts $\hat{f} : \mathcal{L}' \to \mathcal{L}$ which lifts f then $g = p^{\mathfrak{c}} M_{\mathrm{st},\hat{f}}(f)$.*
(2) *let $f' : T'' \to T'$ be a morphism in $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},r}$ with $\mathcal{L}''$ the lift of T'' and $g' : M_{st,\mathcal{L}'}(T') \to M_{\mathrm{st},\mathcal{L}''}(T'')$ the morphism in $M_{\mathrm{tor}}(\varphi, N, \Gamma)$ attached to f', $\mathcal{L}'$ and $\mathcal{L}''$. If there exists a morphism of lifts $\hat{h} : \mathcal{L}'' \to \mathcal{L}$ which lifts $f \circ f'$ then $g' \circ g = p^{2\mathfrak{c}} M_{\mathrm{st},\hat{h}}(f \circ f')$.*

If $er < p-1$ then $\mathfrak{c} = 0$. $M_{\mathrm{st},\mathcal{L}}(T)$ is independent on the choice of lift and $M_{\mathrm{st},\mathcal{L}}$ is a functor from $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},r}$ to $M_{\mathrm{tor}}(\varphi, N, \Gamma)$.

Corollary 3.1.2. *Notations as above, assume that $f : T' \to T$ is an isomorphism and $f' = f^{-1} : T \to T'$ is the inverse map. Then $g' \circ g|_{M_{\mathrm{st},\mathcal{L}}(T)} = p^{2\mathfrak{c}} \mathrm{Id}|_{M_{\mathrm{st},\mathcal{L}}(T)}$ and $g \circ g'|_{M_{\mathrm{st},\mathcal{L}'}(T')} = p^{2\mathfrak{c}} \mathrm{Id}|_{M_{\mathrm{st},\mathcal{L}'}(T')}$.*

Remark 3.1.3. It is natural to ask if one can define a reasonable filtration structure inside $M_{\mathrm{st},\mathcal{L}}(T)$ such that the above theorem is valid for such filtration (together with other structures). Though the answer is positive the proof is too complicated to include here. See the forthcoming work [Liu09a].

To proceed the proof, we study torsion representations arising from a torsion version of $(\varphi, \hat{G})$-modules [6], and show that the p-adic Hodge structures in $M_{\mathrm{st},\mathcal{L}}(T)$ are encoded in those structures. For this, we need the input from torsion Kisin modules. So we recall some facts for torsion Kisin modules, and refer [Liu07b], §2 and §3 for more details.

3.2. **Torsion Kisin modules.** In the sequel, for any $\mathbb{Z}$-module M, we denote M/p^nM by M_n. Recall $'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$ denotes the category of φ-modules of height r. Let $\mathfrak{M} \in {}'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$, we call $\mathfrak{M}$ a *torsion Kisin module* (of height r) if $\mathfrak{M}$ is killed by some p-power and there exists an injection $\mathfrak{L}' \hookrightarrow \mathfrak{L}$ in $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that $\mathfrak{M} \simeq \mathfrak{L}/\mathfrak{L}'$. We denote $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ the full subcategory of torsion Kisin modules of height r in $'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$. The following lemma summarizes some useful facts on (torsion) Kisin modules.

Lemma 3.2.1. (1) *The following statements are equivalent:*
(a) *$\mathfrak{M}$ is a torsion Kisin module.*
(b) *$\mathfrak{M} \in {}'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$, $\mathfrak{M}$ is killed by some p-power and u-torsion free.*
(c) *$\mathfrak{M} \in {}'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$ and $\mathfrak{M}$ is a successive extension of finite free $k[\![u]\!]$-modules $\mathfrak{M}_i$ with $\mathfrak{M}_i \in {}'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$.*

(2) *Let $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ be an exact sequence of φ-modules. Suppose that $\mathfrak{M}'$, $\mathfrak{M}'$, $\mathfrak{M}''$ are u-torsion free and $\mathfrak{M} \in {}'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$. Then $\mathfrak{M}'$, $\mathfrak{M}'' \in {}'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$.*

Proof. See Lemma 2.3.2 and Proposition 2.2.3 in [Liu07b]. □

Now combining the above results with the following input, we are ready to prove Lemma 2.5.3.

Lemma 3.2.2. *Assume that $er < p-1$. Let $\mathfrak{M}$, $\mathfrak{M}' \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. If $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \simeq \mathfrak{M}' \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ as φ-modules. Then $\mathfrak{M} \simeq \mathfrak{M}'$.*

Proof. If $er < p-1$ then the minimal model and maximal model coincides. See Remark after Corollary 3.2.6 in [CL09]. □

Proof of Lemma 2.5.3. (1) By Lemma 3.2.1 (1), $\mathfrak{M}$ is a torsion Kisin module and $\mathfrak{M}$ is u-torsion free. Assume that $\mathfrak{M}$ is killed by p^n. We prove the statement by induction on n. If $n = 1$, since $\mathfrak{M}$ is u-torsion free then $\mathfrak{M}$ is a finite free $\mathfrak{S}_1$-module. Suppose that the statement holds for $n-1$. Consider the exact sequence $0 \to p^{n-1}\mathfrak{M} \to \mathfrak{M} \to \mathfrak{M}_{n-1} \to 0$. We claim that $\mathfrak{M}_{n-1}$ is a Kisin module killed by p^{n-1}. To see the claim, let $\mathcal{K}$ be the kernel of map $q : \mathfrak{M} \to \mathfrak{M}_{n-1} \to \mathfrak{M}_{n-1} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. We get an exact sequence of u-torsion free φ-modules: $0 \to \mathcal{K} \to \mathfrak{M} \to q(\mathfrak{M}) \to 0$. By Lemma 3.2.1 (2), we see that $\mathcal{K}$ and $q(\mathfrak{M})$ are torsion Kisin modules. It is easy to check that $\mathcal{K} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \simeq p^{n-1}\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. Hence $\mathcal{K} = p^{n-1}\mathfrak{M}$ by Lemma 3.2.2, and then $\mathfrak{M}_{n-1} = q(\mathfrak{M})$ is a torsion Kisin module.

By induction, $\mathfrak{M}_{n-1} \simeq \mathfrak{S}_{n_1}\bar{e}_1 \oplus \cdots \oplus \mathfrak{S}_{n_{d'}}\bar{e}_{d'} \oplus \cdots \oplus \mathfrak{S}_{n_d}\bar{e}_d$ with $e_i \in \mathfrak{M}$ and $\bar{e}_i$ the image of e_i under the natural map $\mathfrak{M} \to \mathfrak{M}_{n-1}$ for $i = 1, \ldots, d$. Let us assume

[6]We will not define torsion (φ, G)-modules in this paper, though we implicitly use many their properties.

$e_1, \ldots, e_{d'}$ are those among e_i killed by p^n but not by p^{n-1}. Set

$$M := \mathfrak{S}_n e_1 \oplus \cdots \oplus \mathfrak{S}_n e_{d'} \oplus \mathfrak{S}_{n_{d'+1}} e_{d'+1} \oplus \cdots \oplus \mathfrak{S}_{n_d} e_d.$$

By Nakayama's lemma, we have the following commutative diagram:

$$(3.2.1)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & p^{n-1}M & \longrightarrow & M & \longrightarrow & M_{n-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \wr & & \\ 0 & \longrightarrow & p^{n-1}\mathfrak{M} & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathfrak{M}_{n-1} & \longrightarrow & 0 \end{array}$$

where the first two columns are surjective and the last one is an isomorphism. Assume that $p^{n-1}e_1, \ldots, p^{n-1}e_{\tilde{d}}$ forms a basis for $p^{n-1}\mathfrak{M}$ with $\tilde{d} < d'$. For any $\tilde{d}+1 \leq i \leq d'$, we have $p^{n-1}e_i = \sum_{j=1}^{\tilde{d}} a_{ij} p^{n-1} e_j$ with $a_{ij} \in \mathfrak{S}$. Now for any $\tilde{d}+1 \leq i \leq d'$, replace e_i by $e_i - \sum_{j=1}^{\tilde{d}} a_{ij} e_j$, and replace M by

$$M := \mathfrak{S}_n e_1 \oplus \cdots \oplus \mathfrak{S}_n e_{\tilde{d}} \oplus \mathfrak{S}_{n-1} e_{\tilde{d}+1} \oplus \cdots \oplus \mathfrak{S}_{n-1} e_{d'} \oplus \mathfrak{S}_{n_{d'+1}} e_{d'+1} \cdots \oplus \mathfrak{S}_{n_d} e_d.$$

We still have a diagram as (3.2.1). The second column is still surjective. Since $p^{n-1}e_i = 0$ for any $\tilde{d}+1 \leq i \leq d'$, the first column is an isomorphism. We easily check that the last column is still an isomorphism. Hence the middle is also an isomorphism $M \simeq \mathfrak{M}$.

(2) Let $M = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. Then $i : \mathfrak{M} \hookrightarrow M$ is an injection. Write $q_n : M \to M_n$ and $\mathfrak{M}_{(n)} = q_n \circ i(\mathfrak{M})$. By Lemma 3.2.1, we see that $\mathfrak{M}_{(n)}$ is a torsion Kisin module. Since M is a finite free $\mathcal{O}_{\mathcal{E}}$-module, M_n is finite $\mathcal{O}_{\mathcal{E},n}$-free. By (1), $\mathfrak{M}_{(n)}$ is finite $\mathfrak{S}_n$-free and $\mathfrak{M}' := \varprojlim_n \mathfrak{M}_{(n)}$ is a finite free Kisin module. It suffices to show that that $\mathfrak{M} = \mathfrak{M}'$ as submodules of M. For any n, set $\mathcal{K}_n = \mathrm{Ker}(q_n)|_{\mathfrak{M}}$. Note that $\mathfrak{M}/\mathcal{K}_n \simeq \mathfrak{M}'_n$. On the other hand, $\mathcal{K}_n = p^n M \cap \mathfrak{M} = p^n \mathfrak{M}' \cap \mathfrak{M}$. By Artin-Rees lemma, the topology defined by $\mathcal{K}_n$ coincides with the p-adic topology on $\mathfrak{M}$. Since $\mathfrak{M}$ is p-adic complete, we see that $\mathfrak{M} = \varprojlim \mathfrak{M}_{(n)} = \mathfrak{M}'$.

□

Recall $\mathrm{Rep}(G_\infty)$ denotes the category of $\mathbb{Z}_p[G_\infty]$-modules of finite $\mathbb{Z}_p$-type. For any $\mathfrak{M} \in \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{tor}}$, we can associate a p-power torsion object in $\mathrm{Rep}(G_\infty)$ via

$$T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p).$$

Proposition 3.2.3. *Let $0 \to L \xrightarrow{j} L' \to T \to 0$ be an exact sequence of $\mathbb{Z}_p[G_\infty]$-modules such that*

(1) *there exists $\mathfrak{L}, \mathfrak{L}' \in \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{fr}}$ such that $T_{\mathfrak{S}}(\mathfrak{L}) \simeq L$ and $T_{\mathfrak{S}}(\mathfrak{L}') \simeq L'$.*
(2) *T is killed by a p-power.*

Then there exists an injection $\mathfrak{j} : \mathfrak{L}' \hookrightarrow \mathfrak{L}$ in $\mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{fr}}$ such that $T_{\mathfrak{S}}(\mathfrak{j}) \simeq j$. Furthermore, let $\mathfrak{M} = \mathfrak{L}/\mathfrak{i}(\mathfrak{L}')$ then $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T$.

Proof. Here we use ingredients on the theory of étale φ-modules as in the proof of Lemma 2.5.4. Since $T_{\mathfrak{S}}$ is fully faithful on the category of finite free Kisin modules, we have a morphism $\mathfrak{j} : \mathfrak{L}' \to \mathfrak{L}$ in $\mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{fr}}$ such that $T_{\mathfrak{S}}(\mathfrak{j}) \simeq j$. To see that $\mathfrak{j}$ is an injection. Applying $\mathrm{Hom}_{G_\infty}(-, \mathcal{O}_{\hat{\mathcal{E}}^{\mathrm{ur}}})$ to the map $j : L \to L'$, we see that

$\mathfrak{j}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}:\mathfrak{L}'\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}\to\mathfrak{L}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$ is an injection. Hence $\mathfrak{j}$ is an injection. Since $\mathfrak{L}$ and $\mathfrak{L}'$ has the same $\mathfrak{S}$-rank, $\mathfrak{M}$ is a torsion Kisin module. Finally, the proof of Lemma 4.4.1 in [Liu07b] showed if we have an exact sequence $0\to\mathfrak{L}'\to\mathfrak{L}\to\mathfrak{M}\to 0$ in $'\mathrm{Mod}^{\varphi,r}_{/\mathfrak{S}}$ with $\mathfrak{L}',\mathfrak{L}\in\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ and $\mathfrak{M}\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ then we have an exact sequence of $\mathbb{Z}_p[G_\infty]$-modules

$$0\to T^\vee_{\mathfrak{S}}(\mathfrak{L}')\to T^\vee_{\mathfrak{S}}(\mathfrak{L})\to T^\vee_{\mathfrak{S}}(\mathfrak{M})\to 0. \tag{3.2.2}$$

Thus $T_{\mathfrak{S}}(\mathfrak{M})\simeq T$. □

Corollary 3.2.4. *Assume that T is in $\mathrm{Rep}^{\mathrm{pst},r}_{\mathrm{tor}}$ and $\mathcal{L}: L\overset{j}{\hookrightarrow}L'$ is a lift of T. Then we have an exact sequence of $W(k)$-modules*

$$0\longrightarrow M_{\mathrm{st}}(L')\xrightarrow{M_{\mathrm{st}}(j)}M_{\mathrm{st}}(L)\longrightarrow M_{\mathrm{st},\mathcal{L}}(T)\longrightarrow 0. \tag{3.2.3}$$

Proof. Restricted to G, by Theorem 2.2.1, there exist $\mathfrak{L}$, $\mathfrak{L}'\in\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{L})\simeq L$ and $T_{\mathfrak{S}}(\mathfrak{L}')\simeq L'$. By the proof of the above Proposition, we get an exact sequence $0\to\mathfrak{L}'\overset{\mathfrak{j}}{\to}\mathfrak{L}\to\mathfrak{M}\to 0$. Since $\mathfrak{M}$ is a torsion Kisin module, $\mathfrak{M}$ is u-torsion free by Lemma 3.2.1. Hence by modulo u, we get an exact sequence: $0\to\mathfrak{L}'/u\mathfrak{L}'\to\mathfrak{L}/u\mathfrak{L}\to\mathfrak{M}/u\mathfrak{M}\to 0$. By the construction of $M_{\mathrm{st},\mathcal{L}}(T)$, we obtain (3.2.3) and $M_{\mathrm{st},\mathcal{L}}(T)\simeq\mathfrak{M}/u\mathfrak{M}$ as φ-modules.

□

The following theorem is an important input for the proof of Theorem 3.1.1.

Theorem 3.2.5. *Let $\mathfrak{M}$ be in $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ or $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. There exists a natural $\mathfrak{S}^{\mathrm{ur}}$-linear injection*

$$\iota_{\mathfrak{S}}:\ \mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}\longrightarrow T^\vee_{\mathfrak{S}}(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{ur}}$$

such that

(1) *$\iota_{\mathfrak{S}}$ is compatible with G_∞-actions and Frobenius on both sides.*
(2) *$\mathfrak{t}^r(T^\vee_{\mathfrak{S}}(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{ur}})\subset\iota_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M})$.*
(3) *If $(\mathfrak{M},\varphi,\hat{G})$ is a finite free $(\varphi,\hat{G})$-module then $\hat{\iota}=W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}}$.*

Proof. See Theorem 3.2.2 in [Liu07b] and Proposition 3.1.3 in [Liu09b]. □

Now let $T\in\mathrm{Rep}^{\mathrm{pst},r}_{\mathrm{tor}}$ and $\mathcal{L}: L\overset{j}{\hookrightarrow}L'$ be a lift of T. Write $\hat{\mathfrak{L}}=(\mathfrak{L},\varphi,\hat{G})$ and $\hat{\mathfrak{L}}'=(\mathfrak{L}',\varphi,\hat{G})$ the $(\varphi,\hat{G})$-modules corresponding to L and L' respectively, and $\mathfrak{j}:\mathfrak{L}'\to\mathfrak{L}$ the corresponding morphism in $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$. Now set $\mathfrak{M}:=\mathfrak{L}/\mathfrak{j}(\mathfrak{L}')$. The proof of Corollary 3.2.4 shows there exists an exact sequence (3.2.3) and $M_{\mathrm{st},\mathcal{L}}(T)\simeq\mathfrak{M}/u\mathfrak{M}$ as φ-modules. By Proposition 3.2.3, we have $T_{\mathfrak{S}}(\mathfrak{M})\simeq T|_{G_\infty}$. Now using the injection $\hat{\iota}$ in (2.2.2), we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L'^{\vee}\otimes_{\mathbb{Z}_p} W(R) & \xrightarrow{W(R)\otimes j^{\vee}} & L^{\vee}\otimes_{\mathbb{Z}_p} W(R) & \longrightarrow & T^{\vee}\otimes_{\mathbb{Z}_p} W(R) & \longrightarrow & 0 \\ & & \Big\uparrow \hat{\iota}_{\hat{\mathfrak{L}}'} & & \Big\uparrow \hat{\iota}_{\hat{\mathfrak{L}}} & & \Big\uparrow \hat{\iota}_{\mathfrak{M}} & & \\ 0 & \longrightarrow & W(R)\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{L}}' & \xrightarrow{W(R)\otimes \mathfrak{j}} & W(R)\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{L}} & \longrightarrow & W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M} & \longrightarrow & 0 \\ & & \Big\downarrow \bmod I_+W(R) & & \Big\downarrow \bmod I_+W(R) & & \Big\downarrow \bmod I_+W(R) & & \\ 0 & \longrightarrow & W(\bar{k})\otimes_{W(k)} M' & \longrightarrow & W(\bar{k})\otimes_{W(k)} M & \longrightarrow & W(\bar{k})\otimes_{W(k)} \bar{M} & \longrightarrow & 0 \end{array} \tag{3.2.4}$$

here $M' = M_{\mathrm{st}}(L')$, $M = M_{\mathrm{st}}(L)$ and $\bar{M} = M_{\mathrm{st},\mathcal{L}}(T)$ and $\hat{\iota}_{\mathfrak{M}}$ is induced from $\hat{\iota}_{\hat{\mathfrak{L}}'}$ and $\hat{\iota}_{\hat{\mathfrak{L}}}$. On the other hand, by Theorem 3.2.5, we have an injection $\iota_{\mathfrak{S},\mathfrak{M}} : \mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M} \hookrightarrow T^{\vee}_{\mathfrak{S}}(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{ur}}$.

Lemma 3.2.6. *$\hat{\iota}_{\mathfrak{M}} = W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S},\mathfrak{M}}$ and $\hat{\iota}_{\mathfrak{M}}$ is an injection.*

Proof. The first statement derives from the facts that $\hat{\iota}_{\hat{\mathfrak{L}}} := W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S},\mathfrak{L}}$ and $\hat{\iota}_{\hat{\mathfrak{L}}'} := W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S},\mathfrak{L}'}$ by Theorem 3.2.5 (3). To see $\hat{\iota}_{\mathfrak{M}}$ is injective, note that $(\varphi(\mathfrak{t}))^r(L^{\vee}\otimes_{\mathbb{Z}_p} W(R)) \subset \hat{\iota}_{\hat{\mathfrak{L}}}(W(R)\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{L}})$ by Theorem 2.2.1. Thus we can define a map $\hat{\iota}^* : L^{\vee}\otimes_{\mathbb{Z}_p} W(R) \to W(R)\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{L}}$ such that $\hat{\iota}^*\circ\hat{\iota}_{\hat{\mathfrak{L}}} = (\varphi(\mathfrak{t})^r)\mathrm{Id}$. $\hat{\iota}^*$ induces a map $\hat{\iota}^*_{\mathfrak{M}} : T^{\vee}\otimes_{\mathbb{Z}_p} W(R) \to W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$ such that $\hat{\iota}^*_{\mathfrak{M}}\circ\hat{\iota}_{\mathfrak{M}} = (\varphi(\mathfrak{t})^r)\mathrm{Id}$. Now it suffices to check that $(\varphi(\mathfrak{t}))^r\mathrm{Id} : W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M} \to W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$ is injective. By Lemma 3.2.1 (3), $\mathfrak{M}$ can be written as a successive extension of finite free $k[\![u]\!]$-modules. Since the functor $\mathfrak{M} \mapsto W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$ is exact on the category of torsion Kisin modules by Lemma 3.1.2 in [CL08], it suffices to check that $(\varphi(\mathfrak{t}))^r\mathrm{Id}$ is injective when $\mathfrak{M}$ is finite $k[\![u]\!]$-free. But $\varphi(\mathfrak{t}) \mod p$ is non-zero in R. So $(\varphi(\mathfrak{t}))^r\mathrm{Id}$ is injective. □

Now let $f : T' \to T$ be in $\mathrm{Rep}^{\mathrm{pst},r}_{\mathrm{tor}}$ and $\mathcal{L}'$ a lift of T' as in Theorem 3.1.1, we have a similar commutative diagram (3.2.4) for $\mathcal{L}'$ and get $\mathfrak{M}' \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ such that $\mathfrak{M}'/u\mathfrak{M}' \simeq M_{\mathrm{st},\mathcal{L}'}(T')$ as φ-modules, $T_{\mathfrak{S}}(\mathfrak{M}') \simeq T'|_{G_\infty}$ and an injection $\hat{\iota}_{\mathfrak{M}'} : W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}' \hookrightarrow T'^{\vee}\otimes_{\mathbb{Z}_p} W(R)$. We use the following result to connect $\mathfrak{M}$ with $\mathfrak{M}'$.

Theorem 3.2.7. *There exists a constant $\mathfrak{c}$ only depending on e and r such that the following holds: for any $\mathfrak{M},\mathfrak{M}' \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ and a morphism $f : T_{\mathfrak{S}}(\mathfrak{M}') \to T_{\mathfrak{S}}(\mathfrak{M})$ in $\mathrm{Rep}(G_\infty)$, there exists a unique morphism $\mathfrak{f} : \mathfrak{M} \to \mathfrak{M}'$ in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{f}) = p^{\mathfrak{c}} f$. If $er < p-1$ then $\mathfrak{c} = 0$.*

Proof. See Theorem 2.4.2 in [Liu07b]. □

Now we are ready to construct g in Theorem 3.1.1. Let $\mathfrak{f}$ be in the above theorem attached to f and set $g := \mathfrak{f} \mod u$, we get a morphism of $W(k)$-modules $g : M_{\mathrm{st},\mathcal{L}}(T) \to M_{\mathrm{st},\mathcal{L}'}(T')$. Since $\mathfrak{f}$ is φ-equivariant, g is φ-equivariant. It takes more complicated arguments in the next subsection to show that g is compatible with other structures. Let us accept the fact that g is a morphism in $M_{\mathrm{tor}}(\varphi, N, \Gamma)$, and prove the remaining statements in Theorem 3.1.1.

Suppose that f can be lifted to $\hat{f}$, then there exists an $\mathfrak{h} : \mathfrak{M} \to \mathfrak{M}'$ such that $T_{\mathfrak{S}}(\mathfrak{h}) \simeq f$ and $M_{\mathrm{st},\hat{f}}(f) \simeq \mathfrak{h} \mod u$. Since $T_{\mathfrak{S}}(\mathfrak{f}) = p^{\mathfrak{c}} f$, by the uniqueness of $\mathfrak{f}$, we have $\mathfrak{f} = p^{\mathfrak{c}}\mathfrak{h}$ and hence $g = p^{\mathfrak{c}} M_{\mathrm{st},\hat{f}}(f)$. Now let $f' : T'' \to T'$ be in $\mathrm{Rep}^{\mathrm{pst},r}_{\mathrm{tor}}$ as in Theorem 3.1.1 (2). By Theorem 3.2.7, we get a morphism $\mathfrak{f}' : \mathfrak{M}' \to \mathfrak{M}''$ such that $T_{\mathfrak{S}}(\mathfrak{f}') = p^{\mathfrak{c}} f'$. Therefore, $T_{\mathfrak{S}}(\mathfrak{f}' \circ \mathfrak{f}) = p^{2\mathfrak{c}}(f \circ f')$. Thus if $f \circ f'$ can be lifted to $\hat{h}$ then there exists an $\mathfrak{h}' : \mathfrak{M} \to \mathfrak{M}''$ in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{h}') = f \circ f'$ and $M_{\mathrm{st},\hat{h}}(f \circ f') = \mathfrak{h}' \mod u$. Using the same argument as above, we see that $\mathfrak{f}' \circ \mathfrak{f} = p^{2\mathfrak{c}}\mathfrak{h}'$. Therefore $g' \circ g = p^{2\mathfrak{c}} M_{\mathrm{st},\hat{h}}(f \circ f')$.

3.3. **Compatibility of Γ and N.** In this subsection, we prove that g is compatible with Γ-action and monodromy operator N. Now consider the following commutative diagram:

$$
\begin{array}{ccccc}
 & & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \xrightarrow{\hat{\iota}_{\mathfrak{M}}} & T^{\vee} \otimes_{\mathbb{Z}_p} W(R) \\
 & \swarrow{\scriptstyle \mathrm{mod}\ I_+W(R)} & \downarrow{\scriptstyle W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{f}} & & \downarrow{\scriptstyle p^{\mathfrak{c}} f^{\vee} \otimes W(R)} \\
W(\bar{k}) \otimes_{W(k)} M & & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' & \xrightarrow{\hat{\iota}_{\mathfrak{M}'}} & T'^{\vee} \otimes_{\mathbb{Z}_p} W(R) \\
\downarrow{\scriptstyle W(\bar{k})\otimes g} & \swarrow{\scriptstyle \mathrm{mod}\ I_+W(R)} & & & \\
W(\bar{k}) \otimes_{W(k)} M' & & & &
\end{array}
\tag{3.3.1}
$$

where $M := M_{\mathrm{st},\mathcal{L}}(T)$ and $M' := M_{\mathrm{st},\mathcal{L}'}(T')$, and $\hat{\iota}_{\mathfrak{M}}$, $\hat{\iota}_{\mathfrak{M}'}$ are injections established in Lemma 3.2.6 and diagram (3.2.4). By diagram (3.2.4), the lift $\mathcal{L}$ induces a natural G_K-action on the $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, which is compatible with the G_K-action on $T^{\vee} \otimes_{\mathbb{Z}_p} W(R)$ via the injection $\hat{\iota}_{\mathfrak{M}}$. Since $p^{\mathfrak{c}} f^{\vee}$ is a morphism of $\mathbb{Z}_p[G_K]$-modules. Thus we see that $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{f}$ is compatible with G_K-actions, not only G_∞-actions. Therefore, g is compatible with G_K-actions and then is compatible with Γ-actions.

To show that g is compatible with N, we use $\hat{N}$ defined in (2.4.5). First note that $\varphi^*\mathfrak{M} := \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is a submodule in $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ because we can devissage $\mathfrak{M}$ into finite free $k[\![u]\!]$-modules, where $\varphi^*\mathfrak{M} \hookrightarrow W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is obvious (here we use the exactness of the functor $\mathfrak{M} \mapsto W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$). Using $\hat{N}$ defined on the lift $\mathfrak{L}' \subset \mathfrak{L}$, we obtain a $W(k)$-linear map $\hat{N}_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. For any $x \in \varphi^*\mathfrak{M}$, by the construction of $\hat{N}_{\mathfrak{M}}$, we see that $\hat{N}_{\mathfrak{M}}(ux) \in I_+W(R)(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$. By modulo $I_+\mathfrak{S}$ and $I_+W(R)$ each side, we get a $W(k)$-linear map $\bar{N}_{\mathfrak{M}} : M \to W(\bar{k}) \otimes_{W(k)} M$.

Lemma 3.3.1. *For any $x \in \varphi^*\mathfrak{M}$, $\tau(x) - x = \varphi(\mathfrak{t})\hat{N}_{\mathfrak{M}}(x)$, and $\bar{N}_{\mathfrak{M}} = N_M$.*

Proof. The first statement easily derives from the construction of $\hat{N}_{\mathfrak{M}}$ and $\hat{N}$ on the lift $\mathfrak{L}' \subset \mathfrak{L}$. For the second statement, consider the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \varphi^*\mathfrak{L}' & \longrightarrow & \varphi^*\mathfrak{L} & \longrightarrow & \varphi^*\mathfrak{M} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{N}} & & \downarrow{\scriptstyle \hat{N}} & & \downarrow{\scriptstyle \hat{N}_{\mathfrak{M}}} & & \\
0 & \longrightarrow & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{L}' & \longrightarrow & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{L} & \longrightarrow & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \longrightarrow & 0
\end{array}
$$

Modulo $I_+\mathfrak{S}$ on the first row and $I_+W(R)$ the second row, By Proposition 2.4.3, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{\mathrm{st}}(L') & \longrightarrow & M_{\mathrm{st}}(L) & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow N & & \downarrow N & & \downarrow \bar{N}_{\mathfrak{M}} & & \\ 0 & \longrightarrow & W(\bar{k}) \otimes_{W(k)} M_{\mathrm{st}}(L') & \longrightarrow & W(\bar{k}) \otimes_{W(k)} M_{\mathrm{st}}(L) & \longrightarrow & W(\bar{k}) \otimes_{W(k)} M & \longrightarrow & 0 \end{array}$$

Note the first and second rows are exact. By the definition of N_M on M, we see that $\bar{N}_{\mathfrak{M}} = N_M$. □

Now to prove that g is compatible with N-actions, it suffices to show the following diagram is commutative:

$$\begin{array}{ccc} \varphi^*\mathfrak{M} & \xrightarrow{\hat{N}_{\mathfrak{M}}} & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \\ \downarrow \varphi^*\mathfrak{f} & & \downarrow W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{f} \\ \varphi^*\mathfrak{M}' & \xrightarrow{\hat{N}_{\mathfrak{M}'}} & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' \end{array}$$

Since $\tilde{\mathfrak{f}} := W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{f}$ is G-compatible, by Lemma 3.3.1, we see for $x \in \varphi^*\mathfrak{M}$,

$$\varphi(\mathfrak{t})\tilde{\mathfrak{f}}(\hat{N}_{\mathfrak{M}}(x)) = \tilde{\mathfrak{f}}(\tau(x) - x) = (\tau - 1)\varphi^*\mathfrak{f}(x) = \varphi(\mathfrak{t})\hat{N}_{\mathfrak{M}'}(\varphi^*\mathfrak{f}(x)).$$

Now it suffices to check that the map $\varphi(\mathfrak{t}) : W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' \to W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'$ induced by multiplying $\varphi(\mathfrak{t})$ is an injection, which has been proved in the proof of Lemma 3.2.6.

3.4. **Representations with coefficients.** Let A be a $\mathbb{Z}_p$-algebra. Let $\mathrm{Rep}_A^{\mathrm{pst},r}$ be the full subcategory of $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{pst},r}$ whose object L is an A-module and the G_K-action and the A-action on L commute. A morphism f in $\mathrm{Rep}_A^{\mathrm{pst},r}$ is a morphism of $A[G_K]$-modules. Let $\mathrm{Rep}_{\mathrm{tor},A}^{\mathrm{pst},r}$ for the full subcategory of $\mathrm{Rep}_{\mathrm{tor}}^{\mathrm{pst},r}$ whose object T can be written as the cokernel of $\mathcal{L} : L \to L'$ with $\mathcal{L}$ a morphism in $\mathrm{Rep}_A^{\mathrm{pst},r}$. So T is an $A[G_K]$-module. We define morphism in $\mathrm{Rep}_{\mathrm{tor},A}^{\mathrm{pst},r}$ to be the morphism of $A[G_K]$-modules. Note that $\mathcal{L}$ is a lift of T. Since $M_{\mathrm{st}}(L)$ and $M_{\mathrm{st}}(L')$ are natural $A \otimes_{\mathbb{Z}_p} W(k)$-modules, $M_{\mathrm{st},\mathcal{L}}(T)$ has an $A \otimes_{\mathbb{Z}_p} W(k)$-module structure induced from those on $M_{\mathrm{st}}(L)$ and $M_{\mathrm{st}}(L')$. The aim of this subsection is to prove the following proposition:

Proposition 3.4.1. *Let $f : T' \to T$ be a morphism in $\mathrm{Rep}_{\mathrm{tor},A}^{\mathrm{pst},r}$ with lifts $\mathcal{L}$, $\mathcal{L}'$ of T, T' in $\mathrm{Rep}_A^{\mathrm{pst},r}$ respectively and $g : M_{\mathrm{st},\mathcal{L}}(T) \to M_{\mathrm{st},\mathcal{L}'}(T')$ the morphism in $M_{\mathrm{tor}}(\varphi, N, \Gamma)$ associated to f constructed in Theorem 3.1.1. Then g is a morphism of $A \otimes_{\mathbb{Z}_p} W(k)$-modules.*

Proof. Let $L \in \mathrm{Rep}_A^{\mathrm{pst},r}$ and $\hat{\mathfrak{L}} := (\mathfrak{L}, \varphi, \hat{G})$ the $(\varphi, \hat{G})$-module attached to $L|_G$. Since $\hat{T}$ is an anti-equivalence, $\mathfrak{L}$ has a structure of A-module. Write $\hat{\mathfrak{L}} := \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{L}$. Since

$$\hat{T}(\hat{\mathfrak{L}}) = \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\hat{\mathfrak{L}}, W(R)) = \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{L}}, W(R)),$$

we see that the injection (2.2.2)

$$\hat{\iota} : W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{L}} \longrightarrow \hat{T}^{\vee}(\hat{\mathfrak{L}}) \otimes_{\mathbb{Z}_p} W(R)$$

is compatible with A-actions on both sides. By diagram (3.2.4), we see that $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ has an A-ction which is compatible with that on $T^\vee \otimes_{\mathbb{Z}_p} W(R)$ via $\hat{\iota}_{\mathfrak{M}}$. Now using diagram (3.3.1), since $p^{\mathfrak{c}} f^\vee \otimes_{\mathbb{Z}_p} W(R)$, $\hat{\iota}_{\mathfrak{M}}$ and $\hat{\iota}_{\mathfrak{M}'}$ are compatible with A-actions, we see that $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{f}$ is compatible with A-actions. Modulo $I_+ W(R)$, we see that g is a morphism of $A \otimes_{\mathbb{Z}_p} W(k)$-modules.

□

4. Applications

In this section, we discuss 2 applications of the above theory. The first one extends the main result in [Liu07a] to the more general setting of potentially semi-stable representations, and the second one is to complete the proof of the compatibility between local Langlands correspondence and Fontaine's construction for Galois representations attached to Hilbert modular forms.

4.1. **Néron-Ogg-Shafarevich criterion of finite level.** Let k be a perfect field of characteristic p, K_0 the fractional field of $W(k)$, K/K_0 a finite totally ramified extension and $G_K := \mathrm{Gal}(\overline{K}/K)$. The aim of this subsection is to prove the following theorem:

Theorem 4.1.1. *Let V be a potentially semi-stable representation of G_K with Hodge-Tate weights in $\{0, \dots, r\}$, and $T \subset V$ a G_K-stable $\mathbb{Z}_p$-lattice. There exists a constant α depending on the dimension $d = \dim_{\mathbb{Q}_p}(V)$, the absolute ramification index $\tilde{e} = [K : K_0]$ and r such that V is semi-stable over K if and only if there exist G_K-stable $\mathbb{Z}_p$-lattices $L' \subset L$ in a semi-stable representation W of G_K with Hodge-Tate weights in $\{0, \dots, r\}$ satisfying $T/p^\alpha T \simeq L/L'$ as $\mathbb{Z}_p[G_K]$-modules.*

Remark 4.1.2. Since crystalline representations with Hodge-Tate weights in $\{0, 1\}$ are equivalent to the representations arising from Barsotti-Tate groups (cf. Theorem (0.3) in [Kis06]), the theorem above is a generalization of the main theorem in [Liu07a] in the case $\mathrm{char}(K) = 0$.

Select a Galois extension K'/K such that V is semi-stable over K' and set $e(K'/K'_0)$ the absolute ramification index of K'. After some finite unramified extension, we can always assume that $K'_0 = K_0$. Put $\Gamma := \mathrm{Gal}(K'/K)$ and $M_a := M_{\mathrm{st}}(T)/p^a M_{\mathrm{st}}(T)$. Then M_a has a natural Γ-action induced from the Γ-action on $M_{\mathrm{st}}(T)$.

Lemma 4.1.3. *Γ acts on $M_{\mathrm{st}}(T)$ trivially if and only if Γ acts on M_b trivially, where $b = 1$ if $p \neq 2$ and $b = 2$ if $p = 2$.*

Proof. In general, let finite group Γ act on a finite free $W(k)$-module M and $\mathrm{Im}(\Gamma)$ be the image of Γ in $\mathrm{Aut}_{W(k)} M$, then we have $\mathrm{Im}(\Gamma)$ injects to $\mathrm{Aut}_{\mathbb{Z}/p^b\mathbb{Z}}(M/p^b M)$. In fact, for any $\gamma \in \Gamma$ fixed, let A be the matrix corresponding to γ. Since Γ is finite, there exists an integer m such that $A^m = I$ with I the identity matrix. On the other hand, if Γ acts on $M/p^b M$ trivially then $A = I \mod p^b$. Using p-adic logarithm, we see that $A = I$. Then Γ acts on M trivially. Therefore, we always have the injection $\mathrm{Im}(\Gamma) \hookrightarrow \mathrm{Aut}_{\mathbb{Z}/p^b\mathbb{Z}}(M/p^b M)$. □

Now take $\alpha = 2\mathfrak{c} + b$ where $\mathfrak{c}$ is the constant constructed in Theorem 3.1.1. Note that $L' \subset L$ is another lift $\mathcal{L}$ of $T/p^\alpha T$. Let $M_{\mathrm{st},\mathcal{L}}(T/p^\alpha T)$ be the (φ, N, Γ)-module defined by the lift $\mathcal{L}$. Note that the Γ-action on $M_{\mathrm{st},\mathcal{L}}(T/p^\alpha T)$ is trivial because W

is semi-stable. By Corollary 3.1.2, there exists morphisms $g : M_{\mathrm{st},\mathcal{L}}(T/p^\alpha T) \to M_\alpha$ and $g' : M_\alpha \to M_{\mathrm{st},\mathcal{L}}(T/p^\alpha T)$ such that $g \circ g' = p^{2\mathfrak{c}}\mathrm{Id}_{M_\alpha}$. Therefore $p^{2\mathfrak{c}}M_\alpha \subset g(M_{\mathrm{st},\mathcal{L}}(T/p^\alpha T))$ and Γ acts on $p^{2\mathfrak{c}}M_\alpha \simeq M_b$ trivially. So by Lemma 4.1.3, Γ acts on $M_{\mathrm{st}}(T)$ trivially and then V is semi-stable over K.

Note that $\mathfrak{c}$ depends on $e(K'/K_0) = e(K'/K)e(K/K_0)$ and r. To prove Theorem 4.1.1, it suffices to show the following result:

Lemma 4.1.4. *Let V be a potentially semi-stable representation of G_K. Then there exists a constant β depending on $d = \dim_{\mathbb{Q}_p} V$ such that there exists a Galois extension K'/K with $e(K' : K) \leq \beta$ and V is semi-stable over K'.*

Proof. Choose a Galois extension K'/K such that V is semi-stable over K'. Replacing K by a finite unramified extension over K, we can assume that K' is totally ramified over K. Let $\Gamma := \mathrm{Gal}(K'/K)$. Γ acts on $D := D_{\mathrm{st}}(T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ which is a finite free K_0-module. Write Γ' the image of Γ in $\mathrm{Aut}_{K_0}(D)$. Note that the Γ-action and the φ-action on D is commutative. Then D with Γ-action descends to a finite free $\mathbb{Q}_p$-module D'. Select a Γ-stable $\mathbb{Z}_p$-lattice $M \subset D'$. Note that $\Gamma' \hookrightarrow \mathrm{Aut}_{\mathbb{Z}/p^b\mathbb{Z}}(M/p^bM) \simeq \mathrm{GL}_d(\mathbb{Z}/p^b\mathbb{Z})$ as discussed in the proof of Lemma 4.1.3. Let β be the size of of $\mathrm{GL}_d(\mathbb{Z}/p^b\mathbb{Z})$ and we are done. □

4.2. **Local Langlands compatibility of Hilbert modular forms.** Let F be a totally real field, $\overline{F}$ the fixed algebraic closure and $G_F := \mathrm{Gal}(\overline{F}/F)$. For any finite prime v of $\mathcal{O}_F$, let G_{F_v} be a decomposition group at v and I_{F_v} the inertia subgroup. Let f be a Hilbert modular eigenform of weight $\underline{k} = (k_1, \dots, k_n)$ with $k_i \geq 2$ all have the same parity, and $\pi = \otimes_v \pi_v$ the algebraic automorphic representation of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$ attached to f.

Let $U = \prod_v U_v \subset \prod_v \mathrm{GL}_2(\mathcal{O}_{F_v})$ be a compact open subgroup, where v runs over finite primes. We assume that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ if π_v is spherical and that f is U-invariant. This last condition can always be satisfied if we replace f by a suitable non-zero vector in π.

Denote by S_0 the set of finite primes where π is spherical. The eigenvalues of operators $T_v = U_v \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} U_v$ and $S_v = U_v \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} U_v$ for $v \in S_0$, acting on f generate a number field E_f. Denote these eigenvalues by t_v and s_v respectively.

Now fix an extension of $\mathfrak{p}$ of the p-adic evaluation of $\mathbb{Q}$ to the $\overline{\mathbb{Q}} \subset \mathbb{C}$, and for any number field $K \subset \overline{\mathbb{Q}}$, we denote $K_{\mathfrak{p}}$ the completion of K with respect to $\mathfrak{p}$. Let S be the subset of S_0 by removing all primes over p. By [Tay89] [7], there exists a continuous representation

$$\rho_{\pi,\mathfrak{p}} : G_F \longrightarrow \mathrm{GL}_2(E_{f,\mathfrak{p}})$$

such that $\rho_{\pi,\mathfrak{p}}$ is unramified at any $v \in S$ and for a *geometric* Frobenius Frob_v at v, the characteristic polynomial of $\rho_{\pi,\mathfrak{p}}(\mathrm{Frob}_v)$ is given by $X^2 - t_vX + \mathbf{N}(v)s_v$. Here $\mathbf{N}(v)$ is the size of the residue field at v.

Now let us return the situation of §2 and use notations there. Suppose that $E/\mathbb{Q}_p$ is a finite extension and V is a finite dimensional E-vector space with continuous

[7] Many mathematicians contributed the construction of $\rho_{\pi,\mathfrak{p}}$, while [Tay89] provided a complete construction.

G_K-action, which makes V into a potentially semi-stable G_K-representation. Set

$$D^*_{\mathrm{pst}}(V) = \varinjlim_L (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_L}$$

where L runs over finite extensions of K. $D^*_{\mathrm{pst}}(V)$ is a $(E \otimes_{\mathbb{Q}_p} \tilde{K}_0)[\mathrm{Gal}(\overline{K}/K)]$-module with an automorphism φ which acts semi-linearly with respect to the $\tilde{K}_0$-action and linearly with respect to the $E[\mathrm{Gal}(\bar{K}/K)]$-action. Following Fontaine, we define a $E \otimes_{\mathbb{Q}_p} \tilde{K}_0$-linear action of Weil-Deligne group WD_K as follows. Let $\tau \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ be the *absolute* Frobenius which sends x to x^p, where $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. Now for an element w of the Weil group W_K, define an integer $\nu(w) \in \mathbb{Z}$ by requiring that w acts on $\bar{k}$ by $\tau^{-\nu(w)}$. For any $g \in W_K$, we let g act on $D^*_{\mathrm{pst}}(V)$ as the product of the commuting operators given by the action of the g and $\varphi^{\nu(w)}$. We extend this to an action of Weil-Deligne group WD_K by letting N act on $D^*_{\mathrm{pst}}(V)$ via its action on B_{st}.

Note that the WD_K-action on $D^*_{\mathrm{pst}}(V)$ descends to a finite $E \otimes_{\mathbb{Q}_p} K_0$-vector space, which we denote by $\sigma(V)$. Let $\sigma^{\mathrm{ss}}(V)$ be the Frobenius semi-simplification of $\sigma(V)$ in the sense that, as a W_K-representation $\sigma^{\mathrm{ss}}(V)$ is the semi-simplification of $\sigma(V)|_{W_K}$, but the the Jordan form of N acting on $\sigma^{\mathrm{ss}}(V)$ is the same as that of N acts on $\sigma(V)$. The aim of this subsection is to prove:

Theorem 4.2.1. *If $\mathfrak{q}|p$ is a prime of F, then $\rho_{\pi,\mathfrak{p}}|_{G_{F_\mathfrak{q}}}$ is potentially semi-stable with p-adic Hodge type corresponding to $\underline{k}$, and $\sigma^{\mathrm{ss}}(\rho_{\pi,\mathfrak{p}}|_{G_{F_\mathfrak{q}}})$ corresponds to $\pi_\mathfrak{q}$ via local Langlands correspondence.*

If π is a discrete series at some finite place, this has been proved by Saito [Sai]. So in the following, we assume that π_v is principal series at all primes $\mathfrak{q}|p$. In this case, the assertion of the theorem is that $\rho_{\pi,\mathfrak{p}}|_{G_{F_\mathfrak{q}}}$ is potentially crystalline with p-adic Hodge type corresponding to $\underline{k}$, in the sense explained in the introduction, and the $W_{F_\mathfrak{q}}$-representation of $\sigma^{\mathrm{ss}}(V_{\pi,\mathfrak{p}})$ is associated to $\pi_\mathfrak{q}$, where $V_{\pi,\mathfrak{p}}$ is the underline space of $\rho_{\pi,\mathfrak{p}}|_{G_{F_\mathfrak{q}}}$.

Let $\sigma_\mathfrak{q}$ denote the $W_{F_\mathfrak{q}}$-representation attached to $\pi_\mathfrak{q}$. We have to show that $V_{\pi,\mathfrak{p}}$ is potentially crystalline and for all $w \in W_{F_\mathfrak{q}}$, we have

$$\mathrm{tr}(w|\sigma^{\mathrm{ss}}(V_{\pi,\mathfrak{p}})) = \mathrm{tr}(w|\sigma_\mathfrak{q}).$$

By Claim 1 in [Sai], it suffices to prove this for $\nu(w) > 0$. Fix such w, there exists a finite solvable extension $F'_\mathfrak{q}/F_\mathfrak{q}$ such that $\sigma_\mathfrak{q}$ is unramified at $W_{F'_\mathfrak{q}}$, $w \in W_{F'_\mathfrak{q}}$ and the image of w in $W_{F'_\mathfrak{q}}/I_{F'_\mathfrak{q}} \simeq \mathbb{Z}$ is a generator.

It follows that after a base change to a finite solvable totally real extension F'/F, we may assume that $\sigma_\mathfrak{q}$ is unramified and that w maps to a generator of $W_{F_\mathfrak{q}}/I_{F_\mathfrak{q}}$. It suffices to show that $V_{\pi,\mathfrak{p}}$ is crystalline with p-adic Hodge type corresponding to $\underline{k}$ and

$$\mathrm{tr}(\varphi^{\nu(w)}|D^*_{\mathrm{st}}(V_{\pi,\mathfrak{p}})) = \mathrm{tr}(w|\sigma_\mathfrak{q}) = t_\mathfrak{q}. \tag{4.2.1}$$

To prove this, we need to recall the construction of $\rho_{\pi,\mathfrak{p}}$ by Taylor. Let $\mathfrak{N}$ be the level of f and $\mathfrak{l}$ a prime does not divide $\mathfrak{N}$. Let $S_{\underline{k}}(\mathfrak{N}, \mathfrak{l})$ be the space of cuspforms of weight $\underline{k}$ and level $\mathfrak{N}\mathfrak{l}$. Write $\mathbb{T}_{\underline{k}}(\mathfrak{N}, \mathfrak{l})$ the $\mathbb{Z}$-algebra in $\mathrm{End}_\mathbb{C}(S_{\underline{k}}(\mathfrak{N}, \mathfrak{l}))$ generated by $T_\mathfrak{q}$ for $\mathfrak{q} \neq \mathfrak{l}$ and $S_\mathfrak{a}$ with $\mathfrak{a}$ prime to $\mathfrak{N}\mathfrak{l}$. We have decomposition $S_{\underline{k}}(\mathfrak{N}, \mathfrak{l}) = S_{\underline{k}}(\mathfrak{N}, \mathfrak{l})^{\mathrm{new}} \oplus S_{\underline{k}}(\mathfrak{N}, \mathfrak{l})^{\mathrm{old}}$. Let us write $\mathbb{T}_{\underline{k}}(\mathfrak{N}, \mathfrak{l})^{\mathrm{new}}$ (resp. $\mathbb{T}_{\underline{k}}(\mathfrak{N}, \mathfrak{l})^{\mathrm{old}}$) the image of $\mathbb{T}_{\underline{k}}(\mathfrak{N}, \mathfrak{l})$ in $\mathrm{End}_\mathbb{C}(S_{\underline{k}}(\mathfrak{N}, \mathfrak{l})^{\mathrm{new}})$ (resp. $\mathrm{End}_\mathbb{C}(S_{\underline{k}}(\mathfrak{N}, \mathfrak{l})^{\mathrm{old}})$). Write $\mathbb{T}'_{\underline{k}}(\mathfrak{N}, \mathfrak{l})^{\mathrm{new}}$

the subalgebra of $\mathbb{T}_{\underline{k}}(\mathfrak{N},\mathfrak{l})^{\mathrm{new}}$ generated by $T_{\mathfrak{q}}$ and $S_{\mathfrak{a}}$ such that $\mathfrak{q}$ and $\mathfrak{a}$ are relatively prime to $\mathfrak{N}\mathfrak{l}$, $\mathbb{T}_{\mathfrak{l}} := \mathbb{T}_{\underline{k}}(\mathfrak{N},\mathfrak{l})^{\mathrm{new}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathbb{T}'_{\mathfrak{l}} = \mathbb{T}'_{\underline{k}}(\mathfrak{N},\mathfrak{l})^{\mathrm{new}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. It turns out that $\mathbb{T}'_{\underline{k}}(\mathfrak{N},\mathfrak{l})^{\mathrm{new}}$ injects into a finite product of some number fields. Thus $\mathbb{T}'_{\mathfrak{l}}$ is inside the product of finite extensions of $\mathbb{Q}_p$. Finally, we denote $\tilde{\mathbb{T}}'_{\mathfrak{l}}$ the product of rings of p-adic integers of these fields. Note that $\mathbb{T}'_{\mathfrak{l}} \subset \tilde{\mathbb{T}}'_{\mathfrak{l}}$.

Theorem 4.2.2 ([Tay89]). *Suppose that $[F:\mathbb{Q}]$ is even. For any positive integer n, there exist infinitely many finite places $\mathfrak{l}_n$ (prime to $\mathfrak{N}$) and ring homomorphism $\lambda_{f,\mathfrak{l}_n} : \mathbb{T}_{\underline{k}}(\mathfrak{N},\mathfrak{l}_n)^{\mathrm{new}} \to \mathcal{O}_{E_f}/\mathfrak{p}^n$ such that the following diagram commutes*

$$\begin{array}{ccccc} & & \mathbb{T}_{\underline{k}}(\mathfrak{N},\mathfrak{l}_n)^{\mathrm{new}} & & \\ & \nearrow & & \searrow^{\lambda_{f,\mathfrak{l}_n}} & \\ \mathbb{T}_{\underline{k}}(\mathfrak{N},\mathfrak{l}_n) & & & & \mathcal{O}_{E_f}/\mathfrak{p}^n \\ & \searrow & & \nearrow & \\ & & \mathbb{T}_{\underline{k}}(\mathfrak{N},\mathfrak{l}_n)^{\mathrm{old}} & & \end{array}$$

where $\mathbb{T}_{\underline{k}}(\mathfrak{N},\mathfrak{l}_n)^{\mathrm{old}} \to \mathcal{O}_{E_f}/\mathfrak{p}^n$ is induced by f.

Lemma 4.2.3. *Let $\mathfrak{l}$ be a finite place of F prime to $\mathfrak{N}p$, then there exists a continuous representation:*

$$\rho_{\mathfrak{l}} : G_F \to \mathrm{GL}_2(\tilde{\mathbb{T}}'_{\mathfrak{l}})$$

such that

(1) *$\rho_{\mathfrak{l}}$ is unramified outside of $\mathfrak{N}\mathfrak{l}p$ and for any $v \nmid \mathfrak{N}\mathfrak{l}p$,*

$$\mathrm{tr}(\rho_{\mathfrak{l}}(\mathrm{Frob}_v)) = T_v, \text{ and } \det(\rho_{\mathfrak{l}}(\mathrm{Frob}_v)) = S_v\mathbf{N}(v).$$

(2) *let $r = \max_i(k_i - 1)$; if $\mathfrak{q} \nmid \mathfrak{l}$ then $\rho_{\mathfrak{l}}|_{G_{F_{\mathfrak{q}}}}$ is a $G_{F_{\mathfrak{q}}}$-stable $\mathbb{Z}_p$-lattice inside a crystalline representation with Hodge-Tate weights in $\{-r,\dots,0\}$, and $\varphi^{\nu(w)}$ on $D^*_{\mathrm{st}}(\rho_{\mathfrak{l}}|_{G_{F_{\mathfrak{q}}}})$ satisfies the equation $X^2 - T_{\mathfrak{q}}X + S_{\mathfrak{q}}\mathbf{N}(\mathfrak{q}) = 0$.*

Proof. We use the same idea of the proof of Lemma 13 in [Bre99]. We have $\mathbb{T}'_{\underline{k}}(\mathfrak{N},\mathfrak{l})^{\mathrm{new}} \subset \prod_g K_g$ where g runs over a basis of eigenforms inside $S_{\underline{k}}(\mathfrak{N},\mathfrak{l})^{\mathrm{new}}$ and K_g is the number field generated by eigenvalues of $T_{\mathfrak{a}}$ and $S_{\mathfrak{a}}$ on g. These forms are "classical", in the sense that one can associate to g a Galois representation $\rho_g : G_F \to \mathrm{GL}_2(K_{g,\mathfrak{p}})$ such that ρ_g satisfies the characterizing property same to that of $\rho_{\pi,\mathfrak{p}}$ (see [Car86]). In particular, in [Sai], Saito proved that $\rho_g|_{G_{F_{\mathfrak{q}}}}$ is potentially semi-stable with Hodge-Tate weights in $\{-r,\dots,0\}$, and $\sigma^{\mathrm{ss}}(\rho_g|_{G_{F_{\mathfrak{q}}}})$ is isomorphic to $\sigma_{\mathfrak{q},g}$ of $\pi_{\mathfrak{q},g}$, where $\pi_g = \otimes_v \pi_{v,g}$ is the automorphic representation attached to g and $\sigma_{\mathfrak{q},g}$ is the Weil-Deligne representation attached to $\pi_{\mathfrak{q},g}$ via local Langlands correspondence. Since $\mathfrak{q} \nmid \mathfrak{N}$ by the fact that $\sigma_{\mathfrak{q}}$ is spherical, $\mathfrak{q} \nmid \mathfrak{N}\mathfrak{l}$. So $\sigma_{\mathfrak{q},g}$ is spherical in our case and we see $\rho_g|_{G_{F_{\mathfrak{q}}}}$ is crystalline and $\varphi^{\nu(w)}$ on $D^*_{\mathrm{st}}(\rho_g|_{G_{F_{\mathfrak{q}}}})$ satisfies $\mathrm{tr}(\varphi^{\nu(w)}) = t_{\mathfrak{q}}(g)$ and $\det(\varphi^{\nu(w)}) = s_{\mathfrak{q}}(g)\mathbf{N}(\mathfrak{q})$, where $t_{\mathfrak{q}}(g)$ and $s_{\mathfrak{q}}(g)$ are the eigenvalues of $T_{\mathfrak{q}}$ and $S_{\mathfrak{q}}$ on g respectively. Hence let $\rho_{\mathfrak{l}} : G_F \to \mathrm{GL}_2(\prod_g K_{g,\mathfrak{p}})$ be the direct sum of ρ_g, we have $\varphi^{\nu(w)}$ on $D^*_{\mathrm{st}}(\rho_{\mathfrak{l}}|_{G_{F_{\mathfrak{q}}}})$ satisfies the equation $X^2 - T_{\mathfrak{q}}X + S_{\mathfrak{q}}\mathbf{N}(\mathfrak{q}) = 0$. Finally, since G_F is compact, we see that $\rho_{\mathfrak{l}}$ factors through $\mathrm{GL}_2(\tilde{\mathbb{T}}'_{\mathfrak{l}})$.

□

We need a little more details on constructing $\rho_{\pi,\mathfrak{p}}$ from [Bre99]. Recall $\lambda_{f,\mathfrak{l}_n} : \mathbb{T}_{\underline{k}}(\mathfrak{N}, \mathfrak{l}_n)^{\mathrm{new}} \to \mathcal{O}_{E_f}/\mathfrak{p}^n$ the ring homomorphism in Theorem 4.2.1. By change of basis, we can assume that $\rho_{\mathfrak{l}_n}(c) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which c is a fixed element of G_F induced from the complex conjugation. Let $r_{\mathfrak{l}_n} = (a_{\mathfrak{l}_n}, d_{\mathfrak{l}_n}, x_{\mathfrak{l}_n})$ be the pseudo-representation attached to $\rho_{\mathfrak{l}_n}$. By Cebotarev density theorem, $r_{\mathfrak{l}_n}$ takes values in $\mathbb{T}'_{\mathfrak{l}_n}$. Set $r_n = \lambda_{f,\mathfrak{l}_n} \circ r_{\mathfrak{l}_n} = (a_n, d_n, x_n)$. By Cebotarev density theorem again, $(r_n)_{n\geq 0}$ forms a compatible system of pseudo-representations with values in $\mathcal{O}_{E_f}/\mathfrak{p}^n$. Then the Galois representation $\rho : G_F \to \mathrm{GL}_2(\mathcal{O}_{E_{f,\mathfrak{p}}})$ attached to $(r_n)_{n\geq 0}$ is just $\rho_{\pi,\mathfrak{p}}$.

Now write L the $G_{F_\mathfrak{q}}$-stable $\mathcal{O}_{E_{f,\mathfrak{p}}}$-lattice (constructed from $(r_n)_{n\geq 0}$ as the above) inside $\rho_{\pi,\mathfrak{p}}$, $E = E_{f,\mathfrak{p}}$ and $A = \mathbb{Z}_p[X, Y]$. For any n, $\mathbb{T}'_{\mathfrak{l}_n}$ is an A-algebra via $X \mapsto T_\mathfrak{q}$ and $Y \mapsto S_\mathfrak{q}$. $\mathcal{O}_E = \mathcal{O}_{E_{f,\mathfrak{p}}}$ is also an A-algebra via $X \mapsto t_\mathfrak{q}$ and $Y \mapsto s_\mathfrak{q}$. Note that $\lambda_{f,\mathfrak{l}_n}$ is a ring homomorphism of A-algebras. Write $V_\mathfrak{l}$ the underline space of $\rho_\mathfrak{l} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for $\rho_\mathfrak{l}$ constructed in Lemma 4.2.3. By Theorem 14 and its proof in [Bre99], for any n sufficient large, there exists a $G_{F_\mathfrak{q}}$-stable $\mathbb{Z}_p$-lattice $L'_{(n)}$ inside $V_{\mathfrak{l}_n} \oplus V_{\mathfrak{l}'_n}$ such that there is a morphism of $\mathbb{Z}_p[G_{F_\mathfrak{q}}]$-modules $g_n : L'_{(n)} \to L/\mathfrak{p}^n L$ satisfying the following two properties:

(1) There exists a constant m_1 only depending on f such that for any n, $\mathfrak{p}^{m_1}(L/\mathfrak{p}^n L) \subset \mathrm{Im}(g_n)$.
(2) g_n is a morphism of A-modules.

Recall $r = \max\{k_1 - 1, \dots, k_g - 1\}$. By Lemma 4.2.3, we see that $L'_{(n)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline with Hodge-Tate weights in $\{-r, \dots, 0\}$. Hence for any n sufficient large, $L/p^n L$ is torsion crystalline with Hodge-Tate weights in $\{-r, \dots, 0\}$. By the main theorem of [Liu07b], $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline with Hodge-Tate weights in $\{-r, \dots, 0\}$.

Now set $N_{(n)} := \mathrm{Im}(g_n) \subset L/\mathfrak{p}^n L$ and $L_{(n)}$ the preimage of $N_{(n)}$ inside L. Note that $\mathfrak{p}^{m_1} L \subset L_{(n)} \subset L$. Since $L/\mathfrak{p}^{m_1} L$ is a finite set, we can select a increasing sequence n_i such that $L_{(n_k)} = L_{(n_j)}$ for all k, j. So to prove (4.2.1), without loss of generality, we can replace L by $L_{(n_i)}$. Note that the natural projection $L \to N_{(n_i)}$ is a morphism of $A[G_{F_\mathfrak{q}}]$-modules. Now $N_{(n_i)}$ has two lifts $q_i : L \to N_{(n_i)}$ and $q'_i : L'_{(n_i)} \to N_{(n_i)}$. Then we have two exact sequences $\mathcal{L}$, $\mathcal{L}'$ of $A[G_{F_\mathfrak{q}}]$-modules:

$$0 \to \tilde{L}_{(n_i)} \to L \to N_{(n_i)} \to 0 \text{ and } 0 \to \tilde{L}'_{(n_i)} \to L'_{(n_i)} \to N_{(n_i)} \to 0,$$

where $\tilde{L}_{(n_i)} = \mathrm{Ker}(q_i)$ and $\tilde{L}'_{(n_i)} = \mathrm{Ker}(q'_i)$. Without loss of generality, we can assume that $\tilde{L}_{(n_i)} \subset p^i L$. Note that the above representations have Hodge-Tate weights in $\{-r, \dots, 0\}$. In order to use integral and torsion theory developed in the last two sections, we take dual and apply functor M_{st}, then we get exact sequences

$$0 \to M_{\mathrm{st}}(\tilde{L}^\vee_{(n_i)}) \to M_{\mathrm{st}}(L^\vee) \to M_{\mathrm{st},\mathcal{L}}(N^\vee_{(n_i)}) \to 0$$

and

$$0 \to M_{\mathrm{st}}(\tilde{L}'^\vee_{(n_i)}) \to M_{\mathrm{st}}(L'^\vee_{(n_i)}) \to M_{\mathrm{st},\mathcal{L}'}(N^\vee_{(n_i)}) \to 0$$

Note that the above sequences are the exact sequences of $A \otimes_{\mathbb{Z}_p} W(k)$-modules, where the A-action on $M_{\mathrm{st},\mathcal{L}}(N^\vee_{(n_i)})$ (resp. $M_{\mathrm{st},\mathcal{L}'}(N^\vee_{(n_i)})$) is induced from that on $M_{\mathrm{st}}(L^\vee)$ (resp. $M_{\mathrm{st}}(L'^\vee_{(n_i)})$). Since $M_{\mathrm{st}}(L^\vee) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = D_{\mathrm{st}}((L \otimes \mathbb{Q}_p)^\vee) = D^*_{\mathrm{st}}(V_{\pi,\mathfrak{p}})$, we see $M := M_{\mathrm{st}}(L^\vee)$ is a φ-stable $W(k)$-lattice inside $D^*_{\mathrm{st}}(V_{\pi,\mathfrak{p}})$. Hence to prove

(4.2.1), it suffices to to show that $\varphi_M^{\nu(w)}$ on M satisfies the following equations:

$$X^2 - t_{\mathfrak{q}}X + s_{\mathfrak{q}}\mathbf{N}(\mathfrak{q}) = 0 \text{ and } \det(X) = s_{\mathfrak{q}}\mathbf{N}(\mathfrak{q}). \tag{4.2.2}$$

Since $\det(\rho_{\pi,\mathfrak{p}}) = \chi\epsilon^{-r}$ corresponds the central character of π via class field theory, where ϵ is the p-adic cyclotomic character and χ is a finite Galois character, we must have $\det(\varphi^{\nu(w)}) = s_{\mathfrak{q}}\mathbf{N}(\mathfrak{q})$. So it suffices to check the first equation of (4.2.2).

By Lemma 4.2.3, $\varphi^{\nu(w)}$ on $M_{\rm st}(L'^{\vee}_{(n_i)})$ satisfies the equation $X^2 - T_{\mathfrak{q}}X + S_{\mathfrak{q}}\mathbf{N}(\mathfrak{q}) = 0$. Hence $\varphi^{\nu(w)}$ on $M_{{\rm st},\mathcal{L}'}(N^{\vee}_{(n_i)})$ satisfies the equation $X^2 - t_{\mathfrak{q}}X + s_{\mathfrak{q}}\mathbf{N}(\mathfrak{q}) = 0$. By Corollary 3.1.2 and Proposition 3.4.1, there exist morphisms $g : M_{{\rm st},\mathcal{L}}(N^{\vee}_{(n_i)}) \to M_{{\rm st},\mathcal{L}'}(N^{\vee}_{(n_i)})$ and $g' : M_{{\rm st},\mathcal{L}'}(N^{\vee}_{(n_i)}) \to M_{{\rm st},\mathcal{L}}(N^{\vee}_{(n_i)})$ of $A \otimes_{\mathbb{Z}_p} W(k)$-modules such that $g' \circ g = p^{2\mathfrak{c}}|_{M_{{\rm st},\mathcal{L}}(N^{\vee}_{(n_i)})}$. Write $H(X) = X^2 - t_{\mathfrak{q}}X + s_{\mathfrak{q}}\mathbf{N}(\mathfrak{q})$. For any $x \in M_{{\rm st},\mathcal{L}}(N^{\vee}_{(n_i)})$,

$$H(\varphi^{\nu(w)})(p^{2\mathfrak{c}}x) = H(\varphi^{\nu(w)})(g' \circ g(x)) = g'(H(\varphi^{\nu(w)})(g(x))) = g'(0) = 0.$$

This means for any $x \in M_{\rm st}(L^{\vee})$, $p^{2\mathfrak{c}}H(\varphi^{\nu(w)})(x) \in M_{\rm st}(\tilde{L}^{\vee}_{(n_i)})$. To prove that $H(\varphi^{\nu(w)}) = 0$ on M, we need to check that $M_{\rm st}(\tilde{L}^{\vee}_{(n_i)}) \subset p^i M_{\rm st}(L^{\vee})$.

Note that $L_{(n_i)} \subset p^i L \subset L$ with quotients killed by some p-power. By Corollary 3.2.4, we have $M_{\rm st}(L^{\vee}_{(n_i)}) \subset M_{\rm st}((p^iL)^{\vee}) \subset M_{\rm st}(L^{\vee})$. It suffices to show that $M_{\rm st}((p^iL)^{\vee}) = p^i M_{\rm st}(L^{\vee})$. Consider the injection $j : L^{\vee} \hookrightarrow (p^iL)^{\vee} = \frac{1}{p^i}L^{\vee}$, and composite by the isomorphism $i : \frac{1}{p^i}L^{\vee} \xrightarrow{p^i} L^{\vee}$, we have $j \circ i = p^i : \frac{1}{p^i}L^{\vee} \to \frac{1}{p^i}L^{\vee}$. Applying functor $M_{\rm st}$ and noting that $M_{\rm st}((p^iL)^{\vee}) = M_{\rm st}(\frac{1}{p^i}L^{\vee})$, we get

$$p^i = M_{\rm st}(i) \circ M_{\rm st}(j) : M_{\rm st}((p^iL)^{\vee}) \xrightarrow{M_{\rm st}(j)} M_{\rm st}(L^{\vee}) \xrightarrow[\sim]{M_{\rm st}(i)} M_{\rm st}((p^iL)^{\vee}).$$

Since $M_{\rm st}(i)$ is an isomorphism, $M_{\rm st}((p^iL)^{\vee}) = p^i M_{\rm st}(L^{\vee})$. This finishes the proof of formula (4.2.1).

It still remains to check that $V_{\pi,\mathfrak{p}}$ has p-adic Hodge type corresponding to $\underline{k}$. We have seen that $V_{\pi,\mathfrak{p}}$ is crystalline with Hodge-Tate weights in $\{-r, \dots, 0\}$. If some k_i is strictly larger than 2, then Blasius and Rogawski proved that $\rho_{\pi,\mathfrak{p}}|_{G_{F_{\mathfrak{q}}}}$ is potentially semi-stable with p-adic Hodge type corresponding to the weight $\underline{k}$ in [BR93]. So in the following, we may assume that $k_i = 2$ for all i. Write $K := F_{\mathfrak{q}}$ and set $D_K := ({\rm B}_{\rm dR} \otimes_{\mathbb{Q}_p} \rho_{\pi,\mathfrak{p}})^{G_{F_{\mathfrak{q}}}}$. Then D_K is an $E \otimes_{\mathbb{Q}_p} K$-module. Without of loss of generality, we may assume that E contains the Galois closure of K. Let Q be the set of all $\mathbb{Q}_p$-embedding $j : K \to \overline{\mathbb{Q}}_p$ of K. We have a natural K-algebra isomorphism $E \otimes_{\mathbb{Q}_p} K \simeq \prod_{j \in Q} E_j$, where E_j is the K-algebra via $j : K \to E$. Hence we have isomorphisms $D_K = \oplus_j D_j$ with $D_j = D_K \otimes_{(E \otimes_{\mathbb{Q}_p} K)} E_j$. Similarly, we have ${\rm Fil}^1 D_K = \oplus_j {\rm Fil}^1 D_j$. It suffices to show that $\dim_{E_j} D_j = 2$ and $\dim_{E_j} {\rm Fil}^1 D_j = 1$. It is easy to see that D_K is a rank-2 finite free $E \otimes_{\mathbb{Q}_p} K$-module. So $\dim_{E_j} D_j = 2$. Let $\{e,\ f\}$ be a $E \otimes_{\mathbb{Q}_p} K$-basis and write $e = \oplus_j e_j$ and $f = \oplus_j f_j$ with $e_j, f_j \in D_j$. Then $\{e_j, f_j\}$ is a basis of D_j. Let $\tilde{D} := \bigwedge_{E \otimes_{\mathbb{Q}_p} K} D_K$ which is rank-1 finite free $E \otimes K$-module with basis $e \wedge f = \oplus_j (e_j \wedge f_j)$. Since $\det(V_{\pi,\mathfrak{p}}) = \chi\epsilon^{-1}$ with χ a finite character, we see that $\tilde{D} = (B_{\rm dR} \otimes_{\mathbb{Q}_p} \det(V_{\pi,\mathfrak{p}}))^{G_{F_{\mathfrak{q}}}}$, ${\rm Fil}^1 \tilde{D} = \tilde{D}$ and ${\rm Fil}^2 \tilde{D} = \{0\}$. Now $\dim_{E_j} {\rm Fil}^1 D_j = 1$ follows the facts that ${\rm Fil}^1 \tilde{D} = {\rm Fil}^1 D_K \wedge D_K$ and ${\rm Fil}^2 \tilde{D} = {\rm Fil}^1 D_K \wedge {\rm Fil}^1 D_K$.

Remark 4.2.4. One can prove that $V_{\pi,\mathfrak{p}}$ has p-adic Hodge type corresponding to $\underline{k}$ only using Taylor's construction of $\rho_{\pi,\mathfrak{p}}$, The idea will be similar to that used above to prove that $\sigma^{\mathrm{ss}}(\rho_{\pi,\mathfrak{p}}|_{G_{F_{\mathfrak{q}}}}) \simeq \pi_{\mathfrak{q}}$. See the forthcoming work [Liu09a].

References

[BR93] Don Blasius and Jonathan D. Rogawski, *Motives for Hilbert modular forms*, Invent. Math. **114** (1993), no. 1, 55–87.

[Bre97] Christophe Breuil, *Représentations p-adiques semi-stables et transversalité de Griffiths*, Math. Ann. **307** (1997), no. 2, 191–224.

[Bre99] ———, *Une remarque sur les représentations locales p-adiques et les congruences entre formes modulaires de Hilbert*, Bull. Soc. Math. France **127** (1999), no. 3, 459–472.

[Car86] Henri Carayol, *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 3, 409–468.

[CF00] Pierre Colmez and Jean-Marc Fontaine, *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), no. 1, 1–43.

[CL08] Xavier Caruso and Tong Liu, *Some bounds for ramification of p^n-torsion semi-stable representations*, Preprint, avaliable at `http://arxiv.org/PS_cache/arxiv/pdf/0805/0805.4227v2.pdf` (2008).

[CL09] ———, *Qausi-semi-stable representations*, Bull. Soc. math. de France **137** (2009), no. 2, 158–223.

[FL82] Jean-Marc Fontaine and Guy Laffaille, *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 4, 547–608 (1983).

[Fon90] Jean-Marc Fontaine, *Représentations p-adiques des corps locaux. I*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 249–309.

[Fon94a] ———, *Le corps des périodes p-adiques*, Astérisque (1994), no. 223, 59–111.

[Fon94b] ———, *Représentations l-adiques potentiellement semi-stables*, Astérisque (1994), no. 223, 321–347, Périodes p-adiques (Bures-sur-Yvette, 1988).

[Fon94c] ———, *Représentations l-adiques potentiellement semi-stables*, Astérisque (1994), no. 223, 321–347, Périodes p-adiques (Bures-sur-Yvette, 1988).

[Fon94d] ———, *Représentations p-adiques semi-stables*, Astérisque (1994), no. 223, 113–184, With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).

[Kis06] Mark Kisin, *Crystalline representations and F-crystals*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 459–496.

[Kis08] ———, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546.

[Liu07a] Tong Liu, *Potentially good reduction of Barsotti-Tate groups*, Journal of Number Theory **126** (2007), no. 2, 115–184.

[Liu07b] ———, *Torsion p-adic Galois representations and a conjecture of fontaine,*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 4, 633–674.

[Liu08] ———, *On lattices in semi-stable representations: a proof of a conjecture of Breuil*, Compos. Math. **144** (2008), no. 1, 61–88.

[Liu09a] ———, *Filtration attached to trosion semi-stable representations*, In preparation (2009).

[Liu09b] ———, *A note on lattices in semi-stable representations*, Preprint, appear at Mathematische Annalen (2009).

[Sai] Takeshi Saito, *Hilbert modular forms and p-adic Hodge theory*, Preprint, available at `http://arxiv.org/PS_cache/arxiv/pdf/0612/0612077v2.pdf`.

[Ski] Christopher Skinner, *A note on the p-adic galois representations attached to Hilbert modular forms*, Preprint.

[Tay89] Richard Taylor, *On Galois representations associated to Hilbert modular forms*, Invent. Math. **98** (1989), no. 2, 265–280.

[Tay95] ———, *On Galois representations associated to Hilbert modular forms. II*, Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), Ser. Number Theory, I, Int. Press, Cambridge, MA, 1995, pp. 185–191.

Department of Mathematics, Purdue University, Indiana, 47907, USA.
E-mail address: tongliu@math.purdue.edu