Quiz 3

$$\text{Let } A = \begin{pmatrix} 1 & s & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(1) Find the value of s such that A is diagonalizable (Hint: For each eigenvalue λ_i, compare multiplicity k_i and $\dim E_{\lambda_i}$, where E_{λ_i} is the λ_i-eigenspace).

Solutions: The characteristic polynomial is

$$P_A(\lambda) = \begin{vmatrix} \lambda - 1 & -s & 1 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2)$$

Hence the eigenvalues of A are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. Since the eigenvalue $\lambda_1 = 1$ has multiplicity 2, A is diagonalizable if and only if the dimension of the 1-eigenspace E_1 is 2. Note that the E_1 is given by the solutions of $(1I_3 - A)X = 0$, namely,

$$\begin{pmatrix} 0 & -s & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We easily see that $z = 0$. So if $s \neq 0$ then $y = 0$ and then E_1 is just spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. In this case, the dimension of E_1 is 1, which is less than the multiplicity 2. Hence E_1 has dimension 2 if and only if $s = 0$, in which case, E_1 has a basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. In summary, A is diagonalizable if and only $s = 0$.

(2) For value s that A is diagonalizable, diagonalize A. Namely, find an invertible matrix P and a diagonal matrix Λ such that $A = P\Lambda P^{-1}$.

Solutions: To diagonalize A, we need find eigenvectors which forms a basis. We have found the basis of E_1 from the above. It suffices to find a basis of E_2, which is the space of the solution for the following system (note $s = 0$ from the above question):

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We easily get a basis $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. So we obtain $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.