EXTRA CREDIT QUESTION 3

Prove the spectral decomposition: Suppose that $A \in \mathbb{R}^{n\times n}$ is a real symmetric matrix. Then there exists an orthogonal matrix Q such that $A = Q^T \Lambda Q$ with Λ a diagonal matrix.

Hint: We can proceed with the following steps:

(1) Show that there exists an orthogonal matrix Q_1 such that

$$Q_1 A Q_1^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & B \end{pmatrix}$$

where B is an $(n-1)\times(n-1)$-matrix. To find Q_1, one need to put the first column of Q_1 to be the unit eigenvector X_1 of the eigenvalue λ_1 (note we have prove in class that λ_1 and X_1 must have real entries). Then we get

$$A = {Q_1}^{-1} \begin{pmatrix} \lambda_1 & * \\ 0 & B \end{pmatrix} Q_1.$$

Since Q_1 is orthogonal, we can conclude that $\begin{pmatrix} \lambda_1 & * \\ 0 & B \end{pmatrix}$ must be also symmetric and then $* = 0$.

(2) Show that B is symmetric.

(3) By induction, $B = Q_2^T \Lambda' Q_2$ with Q_2 an orthogonal matrix and Λ' a diagonal matrix.

(4) Show that $Q = \begin{pmatrix} 1 & 0 \\ 0 & Q_2 \end{pmatrix} Q_1$ is the required matrix and finish the induction.