Math 265, Practice Midterm 1

Sept 23, 2015

Name: ______________________________

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.
3. You may use one 4-by-6 index card, both sides.

Score		
1	15	
2	15	
3	15	
4	15	
5	20	
6	10	
7	10	
Total	100	

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

(a) If A^2 makes sense then A is a square matrix.

(b) A system of linear equations can not have exactly 2 solutions.

(c) If $AB = AC$ and $A \neq 0$ then $B = C$.

(d) Let W be a subspace of a vector space V. If $v \in W$ then $-v \in W$.

(e) $\det(2A) = 2\det(A)$.

	(a)	(b)	(c)	(d)	(e)
Answer	T	T	F	T	F

2. Quick Questions, $A,\ B,\ C,\ X,\ b$ are always matrices here:

(a) Suppose that $\det(A) = \det(A^{-1})$. Find $\det(A)$.

Solutions: Since $\det(A) = \det(A)^{-1}$, $\det(A)^2 = 1$. So $\det(A) = \pm 1$.

(b) Suppose $AX = 2X$ and $A^3X = aX$. Then $a =$?

Solutions: $a = 8$

(c) Is the set $\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 | x, y \text{ are integers}\}$ a subspace of $\mathbb{R}^2$, why?

Solutions: Let $c = 0.5$ and $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $cu = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ is not in the set. Hence the set is not closed under scalar multiplication and it is *not* a subspace of $\mathbb{R}^2$.

3. Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \end{bmatrix}.$$

(a) Compute $2I_2 - AB^T$.

Solutions:

$$2I_2 - AB^T = \begin{bmatrix} 3 & -3 \\ -7 & 8 \end{bmatrix}$$

(b) Is $2I_2 - AB^T$ invertible? If it is then find the inverse.

Solutions:

$$(2I_2 - AB^T)^{-1} = \frac{1}{3} \begin{bmatrix} 8 & 3 \\ 7 & 3 \end{bmatrix}$$

4. Consider the following linear system

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & a^2-4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ a \end{pmatrix}$$

(a) Determine all values of a such that the system has no solution.

Solutions: The augmented matrix of the above system is

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 7 \\ 1 & 1 & a^2-4 & a \end{pmatrix}.$$

Then we get the echelon form:

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & a^2-4 & a-2 \end{pmatrix}.$$

Then the system has no solution if and only if $a^2 - 4 = 0$ and $a - 2 \neq 0$. So $a = -2$

(b) Determine all values of a such that the system has infinitely many solutions.

Solutions: The system has infinitely many solutions if and only if $a^2 - 4 = a - 2 = 0$. So $a = 2$.

(c) Determine all values of a such that the system has a unique solution.

Solutions: The system has unique solution if and only if $a^2 - 4 \neq 0$. That is $a \neq \pm 2$.

5. (a) Compute

$$\begin{vmatrix} -2 & 0 & 0 & 0 \\ 5 & 3 & 5 & 7 \\ 3 & 0 & 2 & 1 \\ 8 & 0 & 2 & 2 \end{vmatrix}.$$

Solution: By cofactor's formula, we get

$$\det(A) = -2 \begin{vmatrix} 3 & 5 & 7 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix} = (-2)3 \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -2 \cdot 3 \cdot 2 = -12.$$

(b) Compute A^{-1}, where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} -3 & -2 & 2 \\ -4 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

6. Solve the following linear system using Cramer's rule.

$$\begin{aligned} -2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 4 \\ -2x_1 - x_2 + x_3 &= -3 \end{aligned}$$

Solutions: By Cramer's rule.

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2$$

$$|A_1| = \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix} = -4$$

$$|A_2| = \begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix} = -6$$

$$|A_3| = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix} = -8$$

So

$$x_1 = \frac{|A_1|}{|A|} = 2,\ x_2 = \frac{|A_2|}{|A|} = 3,\ x_3 = \frac{|A_3|}{|A|} = 4.$$

7. Let V be the set of all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $abcd = 0$. Let operation $\oplus$ be the standard addition of matrices and the operation $\odot$ be the standard scalar multiplication of matrices.

(a) Is V closed under addition ?

Solution: No. Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Both A and B are in V. But $A + B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is not in V.

(b) Is V closed under scalar multiplication?

Solution: Yes. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in V. By definition $abcd = 0$. Then for any $x \in \mathbb{R}$, $xA = \begin{pmatrix} xa & ab \\ xc & xd \end{pmatrix}$. We get $xaxbxcxd = x^4abcd = 0$. So cA is in V.

(c) What is zero vector in the set V?

Solutions: It is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(d) Does every matrix A in V have a negative matrix that is in V? Explain.

Solutions: Yes, If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the negative vector $-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ is still inside V because $(-a)(-b)(-c)(-d) = adcd = 0$.

(e) Is V a vector space? Explain.

Solutions: No, it is not a vector space because it is not closed under addition.