Math 265, Midterm 1

October 2nd, 2015

Name: ______________________________

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.
3. You may use one 4-by-6 index card, both sides.

Score		
1	15	
2	15	
3	15	
4	15	
5	20	
6	10	
7	10	
Total	100	

1. The following are true/false questions. You don't have to justify your answers. Just write down either **T** or **F** in the table below. A, B, C, X, b are always matrices here.

(a) If A is symmetric then A is a square matrix.

(b) If $AB = AC$ and A is invertible then $B = C$.

(c) If a system of linear equations has 2 solutions then it has infinitely many solutions.

(d) Let L be a straight line in the plane $\mathbb{R}^2$. Then L is a subspace of $\mathbb{R}^2$.

(e) If $A^2 = 0$ then A^{-1} does not exist.

	(a)	(b)	(c)	(d)	(e)
Answer	**T**	**T**	**T**	**F**	**T**

2. Quick Questions (no details of explanation needed), A, B, C, X, b are always matrices here:

(a) Suppose that $A^2 = A$. $\det(A) =?$

Solutions: We have $\det(A)^2 = \det(A)$. So $\det(A) = 0$ or $\det(A) = 1$.

(b) Suppose A is a 3×3-matrix and $\det(A) = 4$ then $\det(2A) =?$

Solutions : $\det(2A) = 2^n \det(A) = 2^3 \cdot 4 = 32$.

(c) Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in \mathbb{R} \text{ and } abcd = 0 \right\}$ be a subset of $\mathbb{R}^{2\times 2}$ (the space of 2 by 2 matrices). Is V a subspace of $\mathbb{R}^{2\times 2}$? Why?

Solutions: No, V is not a subspace of $\mathbb{R}^{2\times 2}$ because it is not closed under addition. For example, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is obvious that A and B are in V. But $A + B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not in V.

3. Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 3 & -2 \end{bmatrix}.$$

(a) Compute $2I_2 - AB^T$.

Solutions:

$$2I_2 - AB^T = \begin{pmatrix} 3 & -3 \\ -5 & 9 \end{pmatrix}.$$

(b) Is $2I_2 - AB^T$ invertible? If it is then find the inverse.

Solutions: Yes, it is invertible because $\det(2I_2 - AB^T) = 12 \neq 0$ and

$$(2I_2 - AB^T)^{-1} = \frac{1}{12}\begin{pmatrix} 9 & 3 \\ 5 & 3 \end{pmatrix}.$$

4. In the following linear system, determine all values of a for which the resulting linear system has

(a) no solution.

(b) a unique solution

(c) infinitely many solutions

$$\begin{aligned} x + y + 4z &= 3 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

Solutions:

The augmented matrix of the above system is

$$\begin{pmatrix} 1 & 1 & 4 & 3 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2 - 5 & a \end{pmatrix}.$$

Then we get the echelon form:

$$\begin{pmatrix} 1 & 1 & 4 & 3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & a^2 - 9 & a - 3 \end{pmatrix}.$$

Then the system has no solution if and only if $a^2 - 9 = 0$ and $a - 3 \neq 0$. So $a = -3$.

The system has unique solution if and only if $a^2 - 9 \neq 0$. That is $a \neq \pm 3$.

The system has infinitely many solutions if and only if $a^2 - 9 = a - 3 = 0$. So $a = 3$.

5. (a) Compute

$$\begin{vmatrix} -2 & 7 & 6 & 8 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 9 & 2 & 2 \end{vmatrix}.$$

Solutions: By cofactor's formula, we get

$$\det(A) = -2 \begin{vmatrix} 0 & 3 & 0 \\ 3 & 2 & 1 \\ 9 & 2 & 2 \end{vmatrix}.$$

Use the cofactor formula again to the first row. We get

$$\det = (-2)(-3) \begin{vmatrix} 3 & 1 \\ 9 & 2 \end{vmatrix} = 6 \cdot -3 = -18.$$

(b) Compute A^{-1}, where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 2 & 5 \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} -3 & -4 & 2 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$$

6. Solve the following linear system using Cramer's rule.

$$\begin{aligned} x_1 \quad &+2x_2 + 3x_3 &= 1 \\ x_1 \quad &+2x_3 &= 0 \\ 2x_1 \quad &+3x_2 - x_3 &= -5 \end{aligned}$$

Solutions: By Cramer's rule.

$$|A| = \begin{vmatrix} 2 & 4 & 6 \\ 1 & 0 & 2 \\ 2 & 3 & -1 \end{vmatrix} = 26$$

$$|A_1| = \begin{vmatrix} 2 & 4 & 6 \\ 0 & 0 & 2 \\ -5 & 3 & -1 \end{vmatrix} = -52$$

$$|A_2| = \begin{vmatrix} 2 & 2 & 6 \\ 1 & 0 & 2 \\ 2 & -5 & -1 \end{vmatrix} = 0$$

$$|A_3| = \begin{vmatrix} 2 & 4 & 2 \\ 1 & 0 & 0 \\ 2 & 3 & -5 \end{vmatrix} = 26$$

So

$$x_1 = \frac{|A_1|}{|A|} = -2, \ x_2 = \frac{|A_2|}{|A|} = 0, \ x_3 = \frac{|A_3|}{|A|} = 1.$$

7. Let V be the set of all positive real numbers. For any $u, v \in V$ and $c \in \mathbb{R}$, define $u \oplus v = uv$ (i.e., $\oplus$ is ordinary multiplication) and define $\odot$ by $c \odot u = u^c$.

(a) Verify that V closed under addition and scalar multiplication.

Solutions: $\forall u, v \in V$, $u \oplus v = uv > 0$. So $u \oplus v \in V$. That is, V is closed under addition.
$\forall c \in \mathbb{R}$, $\forall u \in V$, $c \odot u = u^c > 0$. So $c \odot u \in V$. That is, V is closed under scalar multiplication.

(b) What is the zero vector in the set V? explain.

Solutions: The zero vector is 1, because $1 \oplus u = 1 \cdot u = u$ for any $u \in V$.

(c) For any $u \in V$, find $-u$.

Solutions: $-u = \frac{1}{u}$, because $\frac{1}{u} \oplus u = \frac{1}{u} \cdot u = 1$ and we know 1 is the zero vector as the above. .