Math 265 Midterm 2

Name: ______________________________

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.
3. You may use one 4-by-6 index card, both sides.

Score		
1	15	
2	15	
3	15	
4	15	
5	20	
6	10	
7	10	
Total	100	

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

(a) Any 5 vectors v_1, v_2, v_3, v_4, v_5 in $\mathbb{R}^4$ are linearly dependent.

(b) Let A be an $m \times n$ matrix. If the rows of A are linear independent, so are the columns.

(c) Consider a system of linear equations $AX = b$ where A is an $n \times n$-matrix. If $\text{rank}(A) = n$ then the system is consistent.

(d) Let V be an inner product space, $W \subset V$ a subspace spanned by $w_1, \ldots, w_m$. Then $v \in W^{\perp}$ if and only if v is orthogonal to $w_1, \ldots, w_m$.

(e) Let A be an $n \times n$-matrix. If $\det(A) = 0$ then the dimension of the column space of A is less than n .

	(a)	(b)	(c)	(d)	(e)
Answer	T	F	T	T	T

2. Quick Questions, $A,\ B,\ C,\ X,\ b$ are always matrices here:

(a) Suppose $u = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Let θ be the angle between u and v. Find $\cos(\theta)$.

Solution:

$$\cos(\theta) = \frac{-1}{\sqrt{12}}$$

(b) Assume that u, v are vectors in a inner product space with the inner product $\langle , \rangle$. Suppose that $\| u \| = 1$, $\| v \| = 2$ with $u \perp v$. Compute $\| u + v \| = \sqrt{\langle u+v, u+v \rangle} = ?$

Solutions:

$$\langle u+v, u+v \rangle = \langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle = 1 + 4 = 5.$$

So $\| u + v \| = \sqrt{5}$.

(c) Find a basis for the space of 3×3 real skew-symmetric matrices (recall that A is skew-symmetric if $A^T = -A$).

Solutions:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

3. (a) For what values of c are the vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ c \end{pmatrix}$ in $\mathbb{R}^3$ linearly independent?

Solutions:

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 0 & 2 & c \end{vmatrix} = 3(c-2).$$

Hence the vectors are linearly independent if and only if $3(c-2) \neq 0$, namely, $c \neq 2$.

(b) Suppose that W is the subspace of $\mathbb{R}^3$ which is spanned by $u = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Find a basis of $W^{\perp}$.

Solutions: $W^{\perp} = \{X \in \mathbb{R}^3 | X \perp u \text{ and } X \perp v\}$. Hence X satisfies the following system

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solve this system, we find that $W^{\perp} = c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. So a basis of $W^{\perp}$ can be $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

4. Let $A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 2 & 2 & -1 & 0 \end{pmatrix}$.

1. Find a basis for the row space of A.

   Solutions: We compute the the reduced echelon form of A is

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   Hence the first two rows $(1\ 1\ 0\ -1), (0\ 0\ 1\ -2)$ forms a basis of the row space of A.

2. Find a basis for the column space of A.

   Solutions: The first and third columns of the reduced echelon form have the leading ones. So the first column and 3rd column $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ of A forms a basis of the column space of A.

3. Find a basis for the null space of A.
   Solutions: From the reduced echelon form, we get equations $x_1 + x_2 - x_4 = 0$ and $x_3 - 2x_4 = 0$. Therefore, we have $x_1 = -x_2 + x_4$ and $x_3 = 2x_4$. Hence, any vector in $N(A)$ can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_2 + x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

   with x_3, x_4 free parameters. So $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ forms a basis of the null space.

4. Verify the equality $\text{rank}(A) + \text{Nullity}(A) = n$.
   Solutions: We find that $\text{rank}(A) = 2$ from (1) and $\text{Nullity}(A) = 2$ from (3). So

$$\text{rank}(A) + \text{Nullity}(A) = 2 + 2 = 4 = n.$$

5. Let $V \subset \mathbb{R}^4$ be a subspace spanned by $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

1. Find an orthonormal basis of V.

   Solutions: Let v_1, v_2 and v_3 denote the above vectors. By Gram-Shmidt process, we compute $w_1 = v_1$ and

   $$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2$$

   as $\langle v_2, w_1 \rangle = 0$. Now

   $$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = v_3 - \frac{1}{2} v_2 - \frac{1}{2} v_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

   .

   Now we get a orthonormal basis $u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, and $u_3 = \frac{w_3}{\|w_3\|} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

2. Let $u = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$. Find $\text{Proj}_V u$ and the distance from u to V.

Solutions: By the formula of projection, we have

$$\text{Proj}_V u = \frac{\langle u, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle u, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \frac{\langle u, w_3 \rangle}{\langle w_3, w_3 \rangle} w_3 = -w_3 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

The distance from u to V is just

$$||u - \text{Proj}_V u|| = 1.$$

6. Consider the following system of linear equation

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= -1 \\ x_1 - 2x_2 &= 0 \end{aligned}$$

1. Find the ranks of the coefficient matrix and the augmented matrix respectively.

   Solutions: Using echelon forms, we see that the rank of coefficient matrix is 2, which the rank of augmented matrix is 3.

2. Is the system consistent? Why?

   Solutions: Since the rank of coefficient matrix is 2, not the rank of augmented matrix is 3. Hence the system is not consistent.

3. If the system is inconsistent then compute the least squares solution.

   Solutions: The least square solution satisfies that $A^T A\hat{X} = A^T b$. Hence we get the system of equations

   $$\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

   So $\hat{x}_1 = \frac{2}{7}$ and $\hat{x}_2 = \frac{3}{7}$.

7. Let A be a 5×3-matrix.

1. What will be the maximal possible rank of A.

   Solutions: Since $\text{rank}(A) \leq m, n$, the maximal rank is 3.

2. Show that rows of A are linearly dependent.

   Proof: The dimension of the row space of A, which is the rank of A, is at most 3 by the above. Now we have 5 rows, 5 rows in a the row space which has the maximal dimension 3 must be linearly dependent.

3. Show that the system $A^T X = 0$ always has a nontrivial solution.

   Proof: A^T is a 3×5-matrix. Now that $\text{nullity}(A^T) + \text{rank}(A^T) = n = 5$. So $\text{nullity}(A) = 5 - \text{rank}(A)$. We have seen that the maximal rank of A^T is at most 3. Hence the nullity of A^T is at least 5-3 = 2. Hence there are infinitely many X satisfies the equation $A^T X = 0$. That is the system $A^T X = 0$ has a nontrivial solution.

   Another proof: As (2), we can see that the columns of A^T are linearly dependent. If we write $\alpha_1, \ldots, \alpha_5$ for columns of A^T then the definition of linearly dependence implies that $\alpha_1, \ldots, \alpha_5$ are linearly dependent if and only if the equation $x_1\alpha_1 + \cdots + x_5\alpha_5 = 0$ has a nontrivial solution. But the equation $x_1\alpha_1 + \cdots + x_5\alpha_5 = 0$ is equivalent to $A^T X = 0$. So $A^T X = 0$ has a nontrivial solution.