QUIZ 3

Find the general solution of the following differential equation

$$y'' - 2y' + y = te^t + 4$$

Solutions:

The characteristic polynomial of the homogeneous equation

$$y'' - 2y' + y = 0$$

is

$$f(r) = r^2 - 2r + 1 = (r-1)^2.$$

Hence the general solution of the homogeneous equation is

$$y = C_1 e^t + C_2 te^t.$$

Now we look for a particular solution by the undetermined coefficients. Consider two equations

$$y'' - 2y' + y = te^t$$

and

$$y'' - 2y' + y = 4$$

For the particular solution $Y_1(t)$ of the first equation, note that $r = 1$ is the double root of $f(r)$. So we need to try $Y_1(t) = t^2(A_1 t + A_0)e^t$. Then we solve that $Y_1(t) = \frac{1}{6}t^3 e^t$.

For the second equation, we just try $Y_2(t) = B$ and it is easy to see that $B = 4$.

So finally, we get the general solution is

$$y(t) = C_1 e^t + C_2 te^t + \frac{1}{6}t^3 e^t + 4.$$

You can also use the method of variation of parameters and the result will be the same. First the Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^{2t}.$$

So the particular solution $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)}dt = -\int \frac{te^t(te^t+4)}{e^{2t}}dt = -\int (t^2+4te^{-t})dt = -\frac{1}{3}t^3+4(t+1)e^{-t}$$

and

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)}dt = \int \frac{e^t(te^t+4)}{e^{2t}}dt = \int t+4e^{-t}dt = \frac{t^2}{2} - 4e^{-t}$$

So $Y(t) = u_1y_1 + u_2y_2 = e^t(-\frac{1}{3}t^3+4(t+1)e^{-t}) + te^t(\frac{t^2}{2} - 4e^{-t}) = \frac{t^3}{6} + 4.$

Then the general solution is

$$y(t) = C_1e^t + C_2te^t + \frac{1}{6}t^3e^t + 4.$$