QUIZ 6

Consider the system of differential equations

$$\mathbf{x}' = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.$$

(1) Find the general solution of the system;

Solutions: The characteristic polynomial of A is $f(r) = r^2 - (1+1)r + (1-4) = r^2 - 2r - 3 = (r-3)(r+1)$. So eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$.

For $\lambda_1 = 3$. Solve $(A - 3I)X = \mathbf{0}$. Note that $A - 3I = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$. So we get eigenvector $\vec{\eta}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

For $\lambda_2 = -1$. Solve $(A - (-1)I)X = \mathbf{0}$. Note that $A + I = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$. So we get eigenvector $\vec{\eta}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Hence the general solution

$$\mathbf{x} = c_1 e^{\lambda_1 t}\vec{\eta}_1 + c_2 e^{\lambda_2 t}\eta_2 = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

(2) Sketch the phase plane;

(3) Determine the stability of the equilibrium solution $\mathbf{x} = \mathbf{0}$ (that is, is $\mathbf{x} = \mathbf{0}$ a saddle point, stable node, or unstable node?).

Solutions: Since $\lambda_1 = 3 > 0$ and $\lambda_2 = -1 < 0$, $\mathbf{x} = \mathbf{0}$ a saddle point.