Math 266, Practice Midterm Exam 2

Name: ______________________________

Ground Rules

1. Calculator is NOT allowed.
2. Show your work for every problem unless otherwise stated (partial credits are available).
3. You may use one 4-by-6 index card, both sides.
4. The table of Laplace transform is available at the last page.

Part I: Multiple Choice (5 points) each. For each of the following questions circle the letter of the correct answer from among the choices given. (No partial credit.)

1. To use the method of undetermined coefficients to find a particular solution of the differential equation

$$y^{(3)} - 3y'' + 3y' - y = 4e^t,$$

which of the following forms of $Y(t)$ should we try?

(a) $Y(t) = Ae^t$
(b) $Y(t) = Ate^t$
(c) $Y(t) = At^2e^t$
(d) $Y(t) = At^3e^t$
(e) $Y(t) = (A_0t^3 + A_1t^4)e^t$

2. A mass weighting 4 lb stretches a spring 1 ft. the mass is attached to viscous damper with a damping constant 2 lb-sec/ft. The mass is pulled down an additional 3 in, and then released. Let $u = u(t)$ denote the displacement of the mass from the equilibrium. (The gravity constant is $g = 32ft/sec^2$.) Then u satisfies the differential equation and initial conditions:

(a) $u'' + 16u' + 32u = 0,\ u(0) = 0,\ u'(0) = 0.$
(b) $u'' + 16u' + 32u = 0,\ u(0) = 0.25,\ u'(0) = 0.$
(c) $4u'' + u' + 32u = 0,\ u(0) = 0.25,\ u'(0) = 0.$
(d) $4u'' + u' + 2u = 0,\ u(0) = 0,\ u'(0) = 0.$
(e) $4u'' + 2u' + 2u = 0,\ u(0) = 0.25,\ u'(0) = 0.$

3. Which of the following forms a fundamental set of solutions to the homogeneous differential equation $y^{(4)} + 2y'' + y = 0$.

(a) $\{\cos t, \sin t, e^t, e^{-t}\}$
(b) $\{e^t, te^t, e^{-t}, te^{-t}\}$
(c) $\{\cos t, t\sin t, t\cos t, \sin t\}$
(d) $\{e^t, e^{-t}\}$
(e) $\{e^t\cos t, e^t\sin t, e^{-t}\cos t, e^{-t}\sin t\}$

4. Suppose the following the initial value problem describes the motion of a certain spring-mass system:

$$4u''(t) + u'(t) + 2u(t) = 3\cos(t),\ \ u(0) = 1,\ \ u'(0) = 0.$$

Then what we can conclude for $\lim_{t\to+\infty} u(t)$?

(a) $\lim_{t\to+\infty} u(t)$ does not exists.

(b) $\lim_{t\to+\infty} = +\infty$.

(c) $\lim_{t\to+\infty} u(t) = -\infty$.

(d) $\lim_{t\to+\infty} u(t) = 0$.

(e) $\lim_{t\to+\infty} u(t) = 1$

5. Consider the function $f(t) = \begin{cases} 1 & 0 \le t < 1, \\ \sin t & 1 \le t \le 2, \\ e^t & t \ge 2. \end{cases}$

   Then $f(t) =$

   (a) $1 + u_1(t)(\sin t - 1) + u_2(t)(e^t - \sin t)$

   (b) $1 + u_1(t)\sin t + u_2(t)e^t$

   (c) $1 + u_1(t)\sin(t-1) + u_2(t)e^{t-2}$

   (d) $1 + u_1(t)(\sin t - 1) + u_2(t)(e^t - \sin t - 1)$

   (e) $1 + u_1(t)(\sin(t-1) - 1) + u_2(t)(e^{t-2} - \sin(t-2))$

Part II: Written answer questions

6. Verify that $y_1(x) = x$ is a solution to

$$x^2 y'' + xy' - y = 0, \quad x > 0.$$

   Then find the general solution of the above equation. (15 points)

7. Consider the differential equation

$$t^2y'' - 2y = 3t^2 - 1, \; t > 0. \tag{1}$$

(a) Show that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ forms a fundamental set of solutions for the homogeneous equation $t^2y'' - 2y = 0$. (10 points)

(b) Find a particular solution of equation (1).(10 points)

8. A mass weighting 4 lb stretches a spring 1.5 in. The mass is displaced 2 in. in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of $2\cos 3t$ lb.

   (a) Formulate the initial value problem describing the motion of the mass.(10 points)

   (b) Find the solution of the initial value problem.(10 points)

   (c) If the given external force is replaced by a force $4\sin\omega t$ of frequency ω, find the value of ω for which resonance occurs.(10 points)

9. Consider the initial value problem

$$y'' - 2y' + 2y = e^t; \quad y(0) = 1, \ y'(0) = 0.$$

(a) Find the Laplace transform $Y(s) = \mathcal{L}(y)$.

(b) Find $y = \mathcal{L}^{-1}(Y(s))$.

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$	2.	$\mathbf{e}^{at}$	$\dfrac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\dfrac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\dfrac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$	8.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\dfrac{2as}{\left(s^2+a^2\right)^2}$	10.	$t\cos(at)$	$\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}$
11.	$\sin(at)-at\cos(at)$	$\dfrac{2a^3}{\left(s^2+a^2\right)^2}$	12.	$\sin(at)+at\cos(at)$	$\dfrac{2as^2}{\left(s^2+a^2\right)^2}$
13.	$\cos(at)-at\sin(at)$	$\dfrac{s\left(s^2-a^2\right)}{\left(s^2+a^2\right)^2}$	14.	$\cos(at)+at\sin(at)$	$\dfrac{s\left(s^2+3a^2\right)}{\left(s^2+a^2\right)^2}$
15.	$\sin(at+b)$	$\dfrac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\dfrac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			