Math 266, Practice Midterm Exam 2

Name: ______________________________

Ground Rules

1. Calculator is NOT allowed.
2. Show your work for every problem unless otherwise stated (partial credits are available).
3. You may use one 4-by-6 index card, both sides.
4. The table of Laplace transform is available at the last page.

Part I: Multiple Choice (5 points) each. For each of the following questions circle the letter of the correct answer from among the choices given. (No partial credit.)

1. To use the method of undetermined coefficients to find a particular solution of the differential equation

$$y^{(3)} - 3y'' + 3y' - y = 4e^t,$$

which of the following forms of $Y(t)$ should we try?

(a) $Y(t) = Ae^t$
(b) $Y(t) = Ate^t$
(c) $Y(t) = At^2e^t$
(d) $Y(t) = At^3e^t$
(e) $Y(t) = (A_0t^3 + A_1t^4)e^t$

2. A mass weighting 4 lb stretches a spring 1 ft. the mass is attached to viscous damper with a damping constant 2 lb-sec/ft. The mass is pulled down an additional 3 in, and then released. Let $u = u(t)$ denote the displacement of the mass from the equilibrium. (The gravity constant is $g = 32ft/sec^2$.) Then u satisfies the differential equation and initial conditions:

(a) $u'' + 16u' + 32u = 0$, $u(0) = 0$, $u'(0) = 0$.
(b) $u'' + 16u' + 32u = 0$, $u(0) = 0.25$, $u'(0) = 0$.
(c) $4u'' + u' + 32u = 0$, $u(0) = 0.25$, $u'(0) = 0$.
(d) $4u'' + u' + 2u = 0$, $u(0) = 0$, $u'(0) = 0$.
(e) $4u'' + 2u' + 2u = 0$, $u(0) = 0.25$, $u'(0) = 0$.

3. Which of the following forms a fundamental set of solutions to the homogeneous differential equation $y^{(4)} + 2y'' + y = 0$.

(a) $\{\cos t, \sin t, e^t, e^{-t}\}$
(b) $\{e^t, te^t, e^{-t}, te^{-t}\}$
(c) $\{\cos t, t\sin t, t\cos t, \sin t\}$
(d) $\{e^t, e^{-t}\}$
(e) $\{e^t\cos t, e^t\sin t, e^{-t}\cos t, e^{-t}\sin t\}$

4. Suppose the following the initial value problem describes the motion of a certain spring-mass system:

$$4u''(t) + u'(t) + 2u(t) = 3\cos(t),\ u(0) = 1,\ u'(0) = 0.$$

Then what we can conclude for $\lim_{t\to+\infty} u(t)$?

(a) $\lim_{t \to +\infty} u(t)$ does not exists.

(b) $\lim_{t \to +\infty} = +\infty$.

(c) $\lim_{t \to +\infty} u(t) = -\infty$.

(d) $\lim_{t \to +\infty} u(t) = 0$.

(e) $\lim_{t \to +\infty} u(t) = 1$

5. Consider the function $f(t) = \begin{cases} 1 & 0 \le t < 1, \\ \sin t & 1 \le t \le 2, \\ e^t & t \ge 2. \end{cases}$

   Then $f(t) =$

   (a) $1 + u_1(t)(\sin t - 1) + u_2(t)(e^t - \sin t)$

   (b) $1 + u_1(t)\sin t + u_2(t)e^t$

   (c) $1 + u_1(t)\sin(t-1) + u_2(t)e^{t-2}$

   (d) $1 + u_1(t)(\sin t - 1) + u_2(t)(e^t - \sin t - 1)$

   (e) $1 + u_1(t)(\sin(t-1) - 1) + u_2(t)(e^{t-2} - \sin(t-2))$

Answer Key: (d), (b), (c), (a), (a).

Part II: Written answer questions

6. Verify that $y_1(x) = x$ is a solution to

$$x^2y'' + xy' - y = 0, \quad x > 0.$$

Then find the general solution of the above equation. (15 points)

Solution: Since $y_1'(x) = x' = 1$, we see that $x^2y'' + xy' - y = x - x = 0$. So $y_1(x) = x$ is a solution of given equation.

To find the second solution, we use the method of *reduction of order*. Assume that $y(x) = y_1(x)v(x)$ is a solution of the equation. Then we found that $v(x)$ satisfies the equation

$$y_1v'' + (2y_1' + p(x)y_1)v' = 0$$

where $p(x) = x/x^2 = 1/x$ here.

Since $y_1(x) = x$, we have the differential equation $xv'' + (2 + \frac{1}{x}x)v' = 0$, that is $xv'' + 3v' = 0$. Hence we solve $v' = Cx^{-3}$. So we find that $v = Cx^{-2} + C_1$.

In particular, we can choose $v(x) = x^{-2}$. So $y(x) = y_1(x)v(x) = x^{-1}$ is a solution of the equation. It is easy to check that the Wronskian

$$W(x, x^{-1}) = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1} \neq 0$$

Hence x and x^{-1} forms a fundamental set of solutions. Thus the general solution is given by

$$y(x) = C_1 x + C_2 \frac{1}{x}.$$

7. Consider the differential equation

$$t^2 y'' - 2y = 3t^2 - 1, \; t > 0. \tag{1}$$

(a) Show that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ forms a fundamental set of solutions for the homogeneous equation $t^2 y'' - 2y = 0$. (10 points)

Solutions: It is standard to check $y_1(t)$ and $y_2(t)$ are solutions of (1) (*Well, you have to show me the details of computations in the exam.*) To check that $y_1(t)$ and $y_2(t)$ forms a fundamental set of solutions, it suffices to check the Wronskian is nonzero. We have

$$W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3 \neq 0$$

(b) Find a particular solution of equation (1).(10 points)

Solutions: Let us find the particular solution by variation of parameters:

$$y(t) = -y_1(t) \int \frac{y_2 g}{W(y_1, y_2)} + y_2(t) \int \frac{y_1 g}{W(y_1, y_2)}.$$

where $g = \frac{3t^2 - 1}{t^2} = 3 - t^{-2}$. From the above, we have known that $W(y_1, y_2) = -3$. Now we get

$$\begin{aligned} y(t) &= -\frac{1}{3}(-t^2 \int t^{-1}(3 - t^{-2}) + t^{-1} \int t^2(3 - t^{-2})) \\ &= \frac{1}{2} + t^2 \ln t - \frac{t^2}{3}. \end{aligned}$$

8. A mass weighting 4 lb stretches a spring 1.5 in. The mass is displaced 2 in. in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of $2\cos 3t$ lb.

   (a) Formulate the initial value problem describing the motion of the mass.(10 points)

   Solution: We have $mg = 4$ and $L = 1.5/12 = 1/8$. Hence $k = mg/L = 32$ and $m = 4/32 = 1/8$. Note the $u(0)$ should be 2 in $= 1/6$ ft. Now we get the initial value problem:

$$\frac{1}{8}u'' + 32u = 2\cos 3t,\ u(0) = \frac{1}{6},\ u'(0) = 0.$$

   (b) Find the solution of the initial value problem.(10 points)
   Solution: The above equation is equivalent to $u'' + 16^2 u = 16\cos 3t$. It is easy to see the characteristic polynomial of the homogenous equation is $r^2 + 16^2$. Then the general solution for the homogenous equation is $C_1 \cos 16t + C_2 \sin 16t$. To find a particular solution, we assume that $Y(t) = A\cos 3t + B\sin 3t$ with $A,\ B$ undetermined coefficients. Then we have $B = 0$ and $A = \frac{16}{16^2-9}$. So the general solution is

$$u(t) = C_1 \cos 16t + C_2 \sin 16t + \frac{16}{16^2 - 9}\cos 3t.$$

   Now by the initial value condition, we get

$$C_1 + \frac{16}{16^2 - 9} = \frac{1}{6},\ \text{and } C_2 = 0.$$

   Then we find the solution of this initial value problem.

$$u(t) = (\frac{1}{6} - \frac{16}{16^2 - 9})\cos 16t + \frac{16}{16^2 - 9}\cos 3t.$$

   (c) If the given external force is replaced by a force $4\sin \omega t$ of frequency ω, find the value of ω for which resonance occurs.(10 points)

   Solution: By the formula, we know resonance occurs when $\omega = \omega_{\max}$ satisfies the following equation

$$\omega_{\max}^2 = \omega_0^2(1 - \frac{\gamma^2}{2mk}).$$

   But we know $\gamma = 0$ in this case. So $\omega_{\max} = \omega_0 = 16$.

9. Consider the initial value problem

$$y'' - 2y' + 2y = e^t; \quad y(0) = 1,\ y'(0) = 0.$$

(a) Find the Laplace transform $Y(s) = \mathcal{L}(y)$.

Solutions: From the above equation, we have

$$\mathcal{L}(y'') - 2\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(e^t) = \frac{1}{s-1}.$$

So

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + 2Y(s) = \frac{1}{s-1}.$$

Plug in $y(0) = 1,\ y'(0) = 0$. We get

$$Y(s) = \frac{s-2}{s^2-2s+2} + \frac{1}{(s-1)(s^2-2s+2)}.$$

(b) Find $y = \mathcal{L}^{-1}(Y(s))$.

Solutions: First note that

$$\frac{s-2}{s^2-2s+2} = \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1}.$$

So

$$\mathcal{L}^{-1}(\frac{s-2}{s^2-2s+2}) = e^t\cos t - e^t\sin t.$$

Assume that

$$\frac{1}{(s-1)(s^2-2s+2)} = \frac{a}{s-1} + \frac{bs+c}{s^2-2s+2}.$$

We solve that $a = 1$, and $b = c = -1$. So

$$\frac{1}{(s-1)(s^2-2s+2)} = \frac{1}{s-1} - \frac{s-1}{(s-1)^2+1}.$$

Then the inverse Laplace transform of the above is $e^t - e^t\cos t$. So

$$y = -e^t\sin t + e^t.$$

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at}, \quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t), \quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			