Math 353 Practice Final Exam

Name: ______________________________

This exam consists of 12 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	16	
2	24	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	100	

Notations: $\mathbb{R}$ denotes the set of real number; F is always a field, for example, $F = \mathbb{R}$; $M_{m\times n}(F)$ denotes the set of $m \times n$-matrices with entries in F; $F^n = M_{n\times 1}(F)$ denotes the set of n-column vectors; $P_n(F)$ denotes the set of polynomials with coefficients in F and the most degree n, that is,

$$P_n(F) = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in F, \ \forall i\}.$$

V is always a finite dimensional vector space over F and T is always a linear operator $T : V \to V$. A^* always denote complex conjugate and transpose of A. For an eigenvalue λ of A, E_λ denotes the λ-eigenspace and K_λ denote the generalized eigenspace.

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. (2 points each)

(a) There is a matrix A with an eigenvalue λ such that the algebraic multiplicity of λ is 5 and $\dim K_\lambda = 4$.

(b) Let $v_1, \ldots, v_m$ are nonzero generalized eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_m$. Suppose that $\lambda_1, \ldots, \lambda_m$ are distinct then $v_1, \ldots, v_m$ are linearly independent.

(c) For an matrix A with all real entries, then all eigenvalues of A are real numbers.

(d) Suppose the Jordan canonical form A has one Jordan block with size > 1. Then A is diagonalizable.

(e) Suppose $W = \text{Span}\{v_1, \ldots, v_n\}$. Then $u \in W^\perp$ if and only if u and v_i are orthogonal for all $i = 1, \ldots, n$.

(f) Let A is be an invertible matrix . Then singular values of A are the same as eigenvalues of A.

(g) If A is a square matrix then $B = A - A^*$ is normal.

(h) If A is unitary then $\det(A) = \pm 1$.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
Answer	F	T	F	F	T	F	T	F

2. Multiple Choice. (3 points each)

(i) Let A be an $m \times n$-matrix. Consider the system of linear equations $AX = b$, which of the following statement is always true:

(a) Suppose $m > n$ then rank of augmented matrix $(A|b)$ can not larger than m.

(b) Suppose $m \leq n$ then $(A^*A)X = A^*b$ always has a unique solution.

(c) Suppose A has full rank then $AX = b$ always has a solution.

(d) If $AX = b$ has a solution then $AX = 0$ has unique solution.

(e) If $AX = b$ has a unique solution then A has to be invertible.

The correct answer is (a).

(ii) Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 3 \\ -2 \\ 2 \end{pmatrix}$. Which vectors from e_1, e_2, e_3, e_4 should be added to v_1, v_2 to form a basis of $\mathbb{R}^4$?

(a) e_1, e_2

(b) e_2, e_3

(c) e_2, e_4

(d) e_1, e_4

(e) such vectors do not exists.

The correct answer is (d).

(iii) Let V be the real vector space of continuous function over $[-1, 1]$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$. Which of the following set is orthonormal?

(a) $1, x, x^2$

(b) $1, e^x$

(c) $1, x, x^2 - \frac{1}{3}$

(d) $\sin x, \cos x$

(e) $\frac{1}{\sqrt{2}}, \frac{\sqrt{3}x}{\sqrt{2}}$.

The correct answer is (e).

(iv) Let

$$A = \begin{pmatrix} 0 & 0 & 0 & a \\ -1 & 0 & 0 & b \\ 0 & -1 & 0 & c \\ 0 & 0 & -1 & d \end{pmatrix}$$

Suppose 0 is an eigenvalue of A with algebraic multiplicity 2. Then which of the following statement is correct?

(a) $a = 0$, $b \neq 0$
(b) $a = b = 0$ and $c \neq 0$
(c) $a = c = 0$ but $b \neq 0$.
(d) $a = d = 0$ but $b \neq 0$.
(e) $b = c = 0$ but $a \neq 0$.

The correct answer is (b).

(v) Suppose $T : V \to V$ be a linear operator with characteristic polynomial $f(t) = t^3 - t^2$. Which of the following statement is always correct?

(a) T is an isomorphism.
(b) T can not be diagonalizable.
(c) For any $v \in V$, $T^3(v) = T^2(v)$.
(d) Such T is unique.
(e) It is possible that T is a unitary operator.

The correct answer is (c).

(vi) Which of the following statement is NOT equivalent that $A \in M_{n\times n}(\mathbb{C})$ is invertible?

(a) Columns of A are linearly independent.
(b) A is normal.
(c) All eigenvalues of A are nonzero.

(d) A has n positive singular values.

(e) The linear system $AX = b$ has unique solution.

The correct answer is (b).

(vii) Let $A = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0.4 & 0.6 \end{pmatrix}$. Which of the following statement is NOT correct?

(a) A is a regular transition matrix.

(b) A is diagonalizable.

(c) All eigenvalues of A are real numbers.

(d) $\lim_{m\to\infty} A^m$ exists.

(e) $\lim_{m\to\infty} A^m = \begin{pmatrix} \frac{1}{3} & 0.5 & 0.5 \\ \frac{1}{3} & 0.5 & 1 \\ \frac{1}{3} & 0 & 0.5 \end{pmatrix}$.

The correct answer is (e).

(viii) Consider the linear system $\begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. Consider the least square solution $\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$ of the above system then $\hat{y} =$

(a) $\frac{1}{5}(\frac{3}{2}a - \frac{3}{2}b - \frac{c}{2} + \frac{d}{2})$.

(b) $\frac{a}{2} + \frac{b}{2} + \frac{c}{2} + \frac{d}{2}$.

(c) $(\frac{3}{2}a - \frac{3}{2}b - \frac{c}{2} + \frac{d}{2})$..

(d) $2a - b + d$.

(e) None of the above answers are correct.

The correct answer is (a).

3. Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$.

(a) Find Jordan canonical form J of A. (5 points)

(b) Find the invertible matrix S so that $A = SJS^{-1}$. (5 points)

Solutions: By cofactor expansion, it is not hard to compute the characteristic polynomial $f_A(t) = t^2(t-1)^2$. Hence we get two distinct eigenvalue $\lambda_1 = 0$ and $\lambda_2 = 1$ both with algebraic multiplicity 2. Now let us compute the K_λ for each λ_i.

For $\lambda_1 = 0$, we have to solve $AX = 0$. We easily see that the eigenspace E_{λ_1} has dimension 1, generated by $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$. Now $A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 3 \\ 2 & 2 & -2 & -2 \end{pmatrix}$.

By solving $A^2X = 0$, we get basis v_1, v_2 of K_{λ_1} where $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Now we build cycle $\{Av_2, v_2\}$, we get $Av_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = v_1$.

Now for $\lambda_2 = 1$, we have $(A - I) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$. The eigenspace has dimension 1 which is spanned by $v_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$. We have

$$(A - I)^2 = \begin{pmatrix} -1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $N(A-I) = K_{\lambda_2}$ is spanned by v_3 and e_4. We build a cycle $(A-I)e_4, e_4$. We easily calculate that $Ae_4 = v_3$.

In summary, the Jordan canonical form of A is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. And

$$S = (v_1, v_2, v_3, e_4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

4. Compute singular value decomposition of $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$. (10 points)

Solutions: To obtain the SVD of A, we first decompose $A * A$.

We first have $A^*A = 2\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$. It is not hard to calculate the characteristic polynomial is $f(t) = -t^2(t-6)$. For $\lambda_1 = 6$, we easily compute that the eigenspace has a basis $v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. For $\lambda_2 = 0$, the eigenspace is defined by equation $x - y + z = 0$. So we get two basis $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. But v_2 and v_3 are not orthogonal. By Gram-Schmidt as in the next problem, we can replace v_3 by $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. By replacing v_i by $v_i/\|v_i\|$. We obtained orthonormal eigenvectors as basis, and hence

$$V = (v_1, v_2, v_3) = \left(\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right)$$

Therefore $A^*A = V\Lambda V^*$ where Λ has $6, 0, 0$ on the main diagonal. So the singular value only has $\sigma_1 = \sqrt{6}$.

Now $u_1 = \frac{1}{\sigma_1} v_1 = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now it suffices to select u_2 such that u_1, u_2 forms an orthonormal basis of $\mathbb{R}^2$. We can select $u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Finally, we get a SVD of A:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^*.$$

5. Let $W \subset \mathbb{R}_3$ be the subapace spanned by $w = (1,1,1)$. Let $W^\perp$ be the orthogonal complement of W. Let $v = (1,0,1)$.

1. Find an orthonormal basis of $W^\perp$. (4 points)
2. Find the projection of v to $W^\perp$. (3 points)
3. Find the $\min\limits_{w \in W^\perp} \|w - v\|$. (3 points)

Solutions : It is easy to check that $W^\perp := \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} | x+y+z+0\}$. We can pick two basis vector of $W^\perp$ to be $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. But v_1 and v_2 is not orthogonal. By Gram-Schmidt, we set $w_1 = v_1$ and

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix}.$$

Replace w_i by $w_i / \|w_i\|$, we obtain an orthonormal basis of $W^\perp$: $u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Now we can use the formula of projection

$$\mathrm{Proj}_{W^\perp} v = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

So the projection is

$$0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Then

$$\min_{w \in W^\perp} \|w - v\| = \|v - \mathrm{Proj}_{W^\perp} v\| = \|\frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\| = \frac{2}{\sqrt{3}}.$$

6. Let A be an $n \times n$-matrix with real entries. Suppose A is skew-symmetric, that is, $A^T = -A$.

(a) Show that if n is odd then $\det(A) = 0$ (3 points)

(b) Show that A is always diagonalizable. (3 points)

(c) Show that eigenvalues of A are either 0 or purely imaginary, that is, $\lambda = bi$ for $b \in \mathbb{R}$. (4 points)

proof

(a) Since $A^T = -A$, we have $\det(A^T) = \det(-A) = (-1)^n \det(A)$. Note that n is odd, $(-1)^n = -1$. So $\det(A) = \det(A^T) = -\det(A)$. That is, $\det(A) = 0$.

(b) Since A is real matrix and $A^T = -A$, we have $A^*A = A^TA = -AA = A(-A) = AA^*$. That is A is normal. So A is diagonalizable.

(c) Since A is normal, A admits spectral decomposition $A = U\Lambda U^*$ where U is a unitary matrix and Λ is a diagonal matrix with eigenvalue λ_i on the main diagonal. Since A is real matrix, we have $A^T = A^* = (U\Lambda U^*)^* = (U^*)^*\Lambda^*U^* = U\Lambda^*U^*$. But $A^T = -A = -U\Lambda U^*$. So $U\Lambda^*U^* = -U\Lambda U^*$. Since U is invertible, we have $\Lambda^* = -\Lambda$. That is, $\bar{\lambda}_i = \lambda_i$ for all i. Then λ_i is either 0 or imaginary.

7. Suppose A and B are square matrices and $AB = BA$.

(a) If v is a generalized eigenvector of A with eigenvalue λ. So is Bv. (5 points)

(b) Suppose that all eigenvalues of A are distinct. Show that there exists an invertible matrix S so that $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$ with Λ_1, Λ_2 being diagonal matrices. (5 points)

Proof:

(a) Let K_λ be the generalized eigenspace of A with the eigenvalue λ and $v \in K_\lambda$. So $(A - \lambda)^p v = \vec{0}$ for some $p > 0$. Now we check that $(A - \lambda I)^p B = B(A - \lambda I)^p$. Indeed, since $AB = BA$,

$$(A - \lambda I)^p B = (A - \lambda I)^{p-1}(A - \lambda I)B = (A - \lambda I)^{p-1}(AB - \lambda B) = \\ (A - \lambda I)^{p-1}(BA - B\lambda I) = (A - \lambda I)^{p-1}B(A - \lambda I).$$

Repeat the above steps, we have $(A - \lambda I)^p B = B(A - \lambda I)^p$. Then $(A - \lambda I)^p Bv = B(A - \lambda)^p v = B\vec{0} = \vec{0}$. That is, $Bv \in K_\lambda$.

(b) Since eigenvalues of A are distinct, A is diagonalizable. So there exists an invertible matrix S such that $A = S\Lambda_1 S^{-1}$ where Λ_1 is a diagonal matrix with distinct eigenvalues λ_i in the main diagonal. Since $AB = BA$, we have $S\Lambda_1 S^{-1}B = BS\Lambda_1 S^{-1}$. Note that S is invertible, this is equivalent to that $\Lambda_1(S^{-1}BS) = (S^{-1}BS)\Lambda_1$. Now we claim that $C = S^{-1}BS = (c_{ij})$ is necessarily a diagonal matrix. In fact $\Lambda_1 C$ is equivalent to times λ_i to the i-th row of C, where $C\Lambda_1$ is equivalent to times λ_j to j-th column of C. Then $\Lambda_1 C = C\Lambda_1$ means that $\lambda_i c_{ij} = \lambda_j c_{ij}$. Since all λ_i are distinct, we conclude that $c_{ij} = 0$ unless $i = j$. That is $S^{-1}BS = C = \Lambda_2$ is a diagonal matrix. So $B = S\Lambda_2 S^{-1}$.

8. Let $A \in M_{m\times n}(\mathbb{C})$. Show the following:

1. $\text{rank}(AA^*) = \text{rank}(A)$. (5 points)
2. If λ is an eigenvalue of AA^* then $\lambda \geq 0$. (5 points).

Proof:

By singular value decomposition, $A = U\Sigma V^*$ where U and V are unitary matrices, and Σ has singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ on the main diagonal, and $r = \text{rank}(A)$. Then

$$AA^* = U\Sigma V^*(U\Sigma V^*)^* = U\Sigma V^* V\Sigma^* U^* = U\Sigma\Sigma^* U^*.$$

It is easy to check that $\Lambda := \Sigma\Sigma^*$ is an $m \times m$-matrix with σ_i^2 on the main diagonal for $i = 1, \ldots, r$. In particular, AA^* is similar to $\Lambda := \Sigma\Sigma^*$ and hence and they share the same eigenvalues and the same rank (note that U is invertible so $\text{rank}(AA^*)\text{rank}(U\Lambda U^*) = \text{rank}(\Lambda)$). Hence AA^*'s eigenvalues λ_i are either σ_i^2 or 0. Hence $\lambda_i \geq 0$. Since $\sigma_i^2 > 0$ has exact $r = \text{rank}(A)$ many on the main diagonal of Λ, $\text{rank}(AA^*) = \text{rank}(\Lambda) = r = \text{rank}(A)$.