Math 353, Practice Midterm 1

Name: ____

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	10	
2	12	
3	15	
4	20	
5	20	
6	23	
Total	100	

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

(a) Let W_1, W_2 be subspaces of a vector space V, then $W_1 \cap W_2$ is a subspace of V.

(b) Let V be a vector space of dimension n. Then any set of m vectors with $m < n$ is linearly independent.

(c) Let $T : V \to W$ be a linear transformation of vector spaces V and W. If $S \subset V$ is a basis of V, then $T(S)$ spans $R(T)$.

(d) Let β be a basis of V and $T : V \to V$ a linear transformation. Write $A = [T]_\beta$. Then T is an isomorphism if and only if $N(A) = \{0\}$.

(e) Let $A \in M_{n\times n}(F)$ be a square matrix. Then the system $AX = b$ is consistent if and only A is invertible.

	(a)	(b)	(c)	(d)	(e)
Answer	T	F	T	T	F

2. Multiple Choice, $A,\ B,\ C,\ X,\ b$ are always matrices here:

(i) Suppose that A is an $m \times n$ matrix with entries in F.

(a) If $\text{rank}(A) = m$ and $n < m$ then the system $Ax = b$ has infinitely many solutions.

(b) If $N(A) = \{0\}$, then $m \geq n$.

(c) If $\text{rank}(A) = n$ and $n \leq m$, then $Ax = b$ has a unique solution.

(d) If $\text{rank}(A) = n$ and $n \leq m$, then $Ax = b$ has more than one solutions.

(e) If $\text{rank}(A) = m$ and $n > m$, then $Ax = b$ has infinitely many solution.

The correct answer is (b) (e).

(ii) Which of the following statement is correct.

(a) Any set of 4 vectors in F^4 is a basis of F^4

(b) Any set of 4 vectors in F^3 is linearly dependent.

(c) Any set of 2 vectors in F^3 is linearly independent.

(d) Any set of 5 vectors in F^4 must span F^4

(e) Any linearly independent subset of F^3 is a basis of F^3

The correct answer is (b).

(iii) Which of the given subsets of $\mathbb{R}_3$ is a subspace?

(a) The set of all vectors of the form (a, b, c) such that $2a + b = c$

(b) The set of all vectors of the form (a, b, c) such that $a + b < c$

(c) The set of all vectors of the form (a, b, c) such that $2a + b = 1$

(d) The set of unit sphere.

(e) The set of all vectors of the form (a, a^2, a^3).

The correct answer is (a).

(iv) Which of the following map is a linear transformation

(a) $L : \mathbb{R}^3 \to \mathbb{R}^2$ by $L(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} x + y \\ y + 1 \end{pmatrix}$.

(b) $T : P_3(\mathbb{R}) \to \mathbb{R}$ by $T(f(x)) = \int_0^1 f(x)^2 dx$

(c) $S : P_3(\mathbb{R}) \to \mathbb{R}$ by $S(f(x) = f'(3)$.

(d) $L : M_{2\times2}(F) \to M_{2\times2}(F)$ by $L(X) = AXA$ where $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

(e) $L : \mathbb{R} \to \mathbb{R}$ by $L(x) = e^x$.

The correct answer is (c)(d).

3. Let $p_1 = 1+x$, $p_2 = x+2x^2-x^3$, $p_3 = 1+2x+2x^2-x^3$, $p_4 = 1+2x+3x^2+x^3$ be polynomials in $P_3(\mathbb{R})$. Find a basis for $\mathrm{Span}\{p_1, p_2, p_3, p_4\} \subset P_3(\mathbb{R})$.

Solutions: Consider equation of vectors $a_1p_1 + a_2p_2 + a_3p_3 + a_4p_4 = 0$ with a_i being unknowns. compare the coefficients of each degree, we arrive the following system of linear equations

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 3 \\ 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Use elementary operation to simplify the equation, we arrive

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then the equation has a solution $a_4 = 0$, $a_1 = 1$, $a_2 = 1$, $a_3 = -1$ So it is clear that p_1, p_2, p_3 are linear dependent. If we remove p_3 or equivalently set $a_3 = 0$. We see that this forces $a_1 = a_2 = a_4 = 0$. So p_1, p_2, p_4 are linearly independent and hence a basis for $\mathrm{Span}\{p_1, p_2, p_3, p_4\}$.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$ and $T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$.

(a) Find $T\begin{pmatrix}2\\1\end{pmatrix}$.

(b) Find the standard matrix representing T.

(c) Find nullity of T and the rank of T.

(d) Given a basis $\beta = \{\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\}$. Find matrix $[T]_\beta$.

Solutions:

(a) Solve equation $\begin{pmatrix}2\\1\end{pmatrix} = x\begin{pmatrix}1\\1\end{pmatrix} + y\begin{pmatrix}1\\2\end{pmatrix}$. We see $x = 3$, $y = -1$. Then

$$T\begin{pmatrix}2\\1\end{pmatrix} = 3T\begin{pmatrix}1\\1\end{pmatrix} - T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}.$$

(b) We find $\begin{pmatrix}1\\0\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix} = -\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}$. So similarly as the above, we have

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}, \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

So the standard matrix representing T is just $A = \begin{pmatrix}2 & 0\\1 & 0\end{pmatrix}$.

(c) Since rows in A ar just multiple to each other, row space of A only has 1 linearly dependent vector. So $\text{rank}(A) = \text{rank}(T) = 1$ and the nullity of T is $2 - \text{rank}(A) = 1$.

(d) Let $Q = \begin{pmatrix}2 & 1\\1 & 0\end{pmatrix}$. Then $[T]_\beta = Q^{-1}AQ = \begin{pmatrix}2 & 1\\4 & 2\end{pmatrix}$.

5. Let $v_1, \dots, v_n \in F^n$ and $A = (v_1, \dots, v_n)$ be the $n \times n$-matrix. Show that $\{v_1, \dots, v_n\}$ is a basis of F^n if and only if A is an invertible matrix via the following steps:

(a) Show that there exists a linear transformation $T : F^n \to F^n$ so that $T(e_i) = v_i$ for all i. Here $\alpha := \{e_1, \dots, e_n\}$ is the standard basis of F^n.

(b) Show that $[T]_\alpha = A$.

(c) Show that if $\{v_1, \dots, v_n\}$ is a basis of F^n then there exists a linear transformation $U : F^n \to F^n$ so that $U(v_i) = e_i$ for all i

(d) Show that $\{v_1, \dots, v_n\}$ is a basis of F^n if and only if A is an invertible matrix.

Proof:

(a) Since $\{e_1, \dots, e_n\}$ is the standard basis of F^n. By Theorem 2.6, there exists unique linear transformation $T : F^n \to F^n$ such that $T(e_i) = v_i$ for all $i = 1, \dots, n$.

(b) Since α is the standard basis of F^n, we always have $v = [v]_\alpha$ for any $v \in F^n$. So

$$[T]_\alpha = ([T(e_1)]_\alpha, \dots, [T(e_n)]_\alpha) = ([v_1]_\alpha, \dots, [v_n]_\alpha) = (v_1, \dots, v_n) = A.$$

(c) If $v_1, \dots, v_n$ is a basis of F^n then we can applies Theorem 2.6 to $\{v_i\}$ and $\{e_i\}$ again, which implies that there exists a unique linear transformation $U : F^n \to F^n$ so that $U(v_i) = e_i$ for all $i = 1, \dots, n$.

(d) Applies Theorem 2.18 to T, we see T is an isomorphism if and only if $A = [T]_\alpha$ is invertible. Now if $v_1, \dots, v_n$ is a basis, then (c) constructed a linear transformation $U : F^n \to F^n$. Note that $UT(e_i) = I_{F^n}(e_i)$ for all i. Using Theorem 2.6 again, UT is necessarily I_{F^n}. Similarly, we see that $TU = I_{F^n}$. So T is invertible and hence $A = [T]_\alpha$ is invertible. Conversely, if A is invertible then T is invertible, which means it is one-to-one and onto. In particular, by Theorem 2.2, $\{v_i = T(e_i),\ i = 1, \dots, n\}$ spans F^n. Since $\{v_1, \dots, v_n\}$ has n vectors, this implies that $\{v_1, \dots, v_n\}$ is a basis of F^n.

6. Let V, W be vector spaces over field F. Let $v_1, \ldots, v_n \in V$ be *linearly independent* vectors.

(a) Show that given $w_1, \ldots, w_n \in W$ then there exists a linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all $i = 1, \ldots, n$.

(b) Could we drop the assumption of linear independence for so that the above statement is still true? why or why not?

(c) Is T necessarily unique, why or why not?

Solutions:

(a) *Proof*: Since v_i are linearly independent, then one can always extend v_i, say $\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_m\}$, to be a basis of V. Now extend w_i to m-vectors $w_1, \ldots, w_n, w_{n+1}, \ldots, w_m$. For example, $w_{n+1} = \cdots = w_m = 0$. Now by Theorem 2.6, there exists a unique linear transformation $T : V \to W$ so that $T(v_i) = w_i$ for all $i = 1, \ldots, m$.

(b) No. We can not drop the assumption that v_i are linearly independent. For example, let $V = W = \mathbb{R}$, $v_1 = v_2 = 1$, but $w_1 = 0$ and $w_2 = 2$. It is not possible to define a linear transformation $T : \mathbb{R} \to \mathbb{R}$ so that $T(v_i) = w_i$.

(c) Contrary to Theorem 2.6, T here in general is not unique as v_i may not form a basis. For example, let $V = W = \mathbb{R}^2$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $w_1 = v_1$. We see $I_V(v_1) = w_1$. On the other hand, consider linear transformation $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$. We still have $T(v_1) = w_1$.