Math 353, Midterm 1

Name: ______________________________

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	15	
2	16	
3	15	
4	15	
5	21	
6	18	
Total	100	

Notations: $\mathbb{R}$ denotes the set of real number; F is always a field, for example, $F = \mathbb{R}$; $M_{m\times n}(F)$ denotes the set of $m \times n$-matrices with entries in F; $F^n = M_{n\times 1}(F)$ denotes the set of n-column vectors; $P_n(F)$ denotes the set of polynomials with coefficients in F and the most degree n, that is,

$$P_n(F) = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in F, \ \forall i\}.$$

Given $A \in M_{m\times n}(F)$, then the linear transformation $L_A : F^n \to F^m$ is defined via $L_A(x) = Ax$ for any $x \in F^n$.

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. (3 points each)

(a) Let W_1, W_2 be subspaces of a vector space V, then $W_1 \cup W_2$ is a subspace of V.

(b) Let $T : V \to W$ be a linear transformation then

$$\text{nullity}(T) + \text{rank}(T) = \dim_F W.$$

(c) Let A, B be $n \times n$-matrices. Then AB is invertible if and only if both A and B are invertible.

(d) Let $A \in M_{n\times n}(F)$ be a square matrix. Consider the linear transformation $L_A : F^n \to F^n$ via $L_A(x) = Ax$. Then for any basis β of F^n, we have $[T]_\beta = A$.

(e) If rows of a matrix A are linearly independent then so are the columns.

	(a)	(b)	(c)	(d)	(e)
Answer	F	F	T	F	F

2. Multiple Choice. (4 points each)

(i) Suppose that A is an $m \times n$ matrix with entries in $\mathbb{R}$ and consider a system of linear equations $Ax = b$ over the field $\mathbb{R}$. Which of the following statement is correct?

(a) If $\text{rank}(A) = m$ and $n > m$ then the system $Ax = b$ has infinitely many solutions.

(b) If $N(A) = \{0\}$, then $m \leq n$.

(c) If $\text{rank}(A) = n$, then $Ax = b$ has a unique solution.

(d) If $\text{rank}(A) = m$ and $n \geq m$, then $Ax = b$ has at least one solution.

(e) If $\text{rank}(A) = m$ and $n = m$, then $Ax = b$ could have no solution.

The correct answer is (a) (d)

(ii) Suppose V is a vector space over F with $\dim_F V = 5$. Which of the following statement is NOT correct.

(a) Any set of 6 vectors in V spans V.

(b) Any set of 6 vectors in V is linearly dependent.

(c) V is isomorphic to F^5.

(d) If a set S of vectors in V is linearly independent then there are at most 5 vectors in S.

(e) Suppose $S = \{v_1, v_2, v_3, v_4, v_5\}$ spans V then S forms a basis of V.

The correct answer is (a)

(iii) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation so that $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$ and $T\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\-1\\4\end{pmatrix}$. Which of the following statement is FALSE?

(a) T is one-to-one.
(b) Such a linear transformation T is unique.
(c) The standard matrix representing T (i.e., the matrix representing T for the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$) is $\begin{pmatrix}1 & -1 & 2\\2 & -2 & 4\end{pmatrix}$.
(d) $\text{rank}(T) = 2$.
(e) $T\begin{pmatrix}-1\\-2\end{pmatrix} = \begin{pmatrix}0\\1\\-2\end{pmatrix}$.

The correct answer is (c)

(iv) Let $A \in M_{n\times n}(F)$ be a square matrix. Which of the following statement is NOT equivalent to the statement that A is invertible.

(a) The linear transformation $L_A : F^n \to F^n$ given by $L_A(x) = Ax$ is an isomorphism.
(b) The system of linear equation $AX = b$ has a unique solution.
(c) The system of linear equation $AX = b$ has a solution.
(d) A is a product of elementary matrices.
(e) Rows of A are linearly independent.

The correct answer is (c)

3. Let $V = M_{2\times 2}(\mathbb{R})$ and $v_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$.

(a) Show that v_1, v_2, v_3 are linearly independent. (7 points)

(b) Find v_4 so that $S = \{v_1, v_2, v_3, v_4\}$ forms a basis of V. (8 points)

Solutions

(a) Consider equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$. Then we have the following system of linear equations

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using elementary (row) operations to simplify the above system, we have

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So it is clear the system only has solution $x_1 = x_2 = x_3 = 0$. So v_1, v_2, v_3 are linearly independent.

(b) Suppose that $v_4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and consider the equation $x_1v_1 + x_2v_2 + x_3v_3 + x_3v_4 = 0$. We get the equation

$$\begin{pmatrix} 1 & 0 & 1 & a \\ 1 & 1 & 2 & b \\ 0 & 2 & 3 & c \\ 0 & -1 & 1 & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using elementary (row) operations to simplify the above system, we have

$$\begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 1 & c-2(b-a) \\ 0 & 0 & 0 & d-2c+5(b-a) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So it is clear that the system only has zero solution if an only if

$$d - 2c + 5(b - a) \neq 0.$$

For example, $v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

4. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation given by $T(f(x)) = f'(x)$. (5 points)

(a) Find the standard matrix representing T, that is, the matrix representing T for the standard basis $\{1, x, x^2\}$. (5 points)

(b) Find nullity of T and the rank of T. (5 points)

(c) Given a basis $\beta = \{1 - x, 1 + x, x^2 + x\}$. Find matrix $[T]_\beta$. (5 points)

Solutions:

(a) We have

$$T(1,\ x,\ x^2) = (0,\ 1,\ 2x^2) = (1,\ x,\ x^2)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So the standard matrix of T is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) It is clear that the range $R(T)$ of T is spanned by $T(1)$, $T(x)$, and $T(x^2)$. Since 1 and $2x$ are linearly independent, they forms of basis of $R(T)$ and hence $\dim_{\mathbb{R}} R(T) = \operatorname{rank}(T) = 2$. By dimension theorem, the nullity of T is $\dim_{\mathbb{R}} P_2(\mathbb{R}) - \operatorname{rank}(T) = 3 - 2 = 1$.

(c) Let $\alpha = \{1, x, x^2\}$ be the standard basis and $Q = [I_{P_2(\mathbb{R})}]_\alpha^\beta$. Since $(1 - x,\ 1 + x,\ x^2 + x) = (1,\ x\ x^2)\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ So we conclude that $Q = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Then

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

5. Let $A \in M_{m\times n}(F)$ be an $m \times n$-matrix and $B \in M_{n\times p}(F)$ be $n \times p$-matrix.

(a) Show that $L_{AB} = L_A L_B$ as linear transformations from F^p to F^m. (5 points)

(b) Show that $R(L_{AB}) \subset R(L_A)$. (5 points)

(c) Show that $\text{rank}(AB) \leq \text{rank}(A)$. (5 points)

(d) Show that $\text{rank}(AB) \leq \text{rank}(B)$ (Hint: use the fact that $\text{rank}(C^t) = \text{rank}(C)$ for any matrix C). (6 points)

Proof:

(a) For any $x \in F^p$, we have $L_{AB}(x) = (AB)x$ and

$$(L_A L_B)(x) = L_A(L_B(x)) = L_A(Bx) = A(Bx) = (AB)x.$$

So $L_{AB} = L_A L_B$.

(b) By definition of range, we have

$$R(L_{AB}) = \{L_{AB}(x) \in F^m | x \in F^p\}.$$

Since $L_{AB} = L_A L_B$ by (a), we have

$$L_{AB}(x) = (L_A L_B)(x) = L_A(L_B(x)) \in R(L_A).$$

Hence $R(L_{AB}) \subset R(L_A)$.

(c) By the definition of rank, we have $\text{rank}(AB) = \dim_F(R(L_{AB}))$ and $\text{rank}(A) = \dim_F R(L_A)$. By (b), we have $R(L_{AB}) \subset R(L_A)$. So $\dim_F R(L_{AB}) \leq \dim_F R(L_A)$ and hence $\text{rank}(AB) \leq \text{rank}(A)$.

(d) Since $\text{rank}(C) = \text{rank}(C^t)$, we have $\text{rank}(AB) = \text{rank}((AB)^t) = \text{rank}(B^t A^t)$. By (c), we have $\text{rank}(B^t A^t) \leq \text{rank}(B^t) = \text{rank}(B)$. Therefore, $\text{rank}(AB) \leq \text{rank}(B)$.

6. Let V, W be vector spaces over a field F and $T : V \to W$ a linear transformation. Let $v_1, \dots, v_n \in V$ be vectors in V.

(a) Suppose that T is one-to-one and $\{v_1, \dots, v_n\}$ are linearly independent. Show that $\{T(v_1), \dots, T(v_n)\}$ are also linearly independent. (6 points)

(b) Suppose that T is onto and $\{v_1, \dots, v_n\}$ spans V. Show that $\{T(v_1), \dots, T(v_n)\}$ spans W. (6 points)

(c) Suppose that T is an isomorphism and $\{v_1, \dots, v_n\}$ is a basis of V. Show that $\{T(v_1), \dots, T(v_n)\}$ is a basis of W. (6 points)

Proof:

(a) To show that $\{T(v_1), \dots, T(v_n)\}$ is linearly independent, we consider vector equation: $x_1T(v_1) + \cdots + x_nT(v_n) = \vec{0}$. It suffices to show that $x_1 = \cdots x_n = 0$. Note that T is linear, we have $T\left(\sum_{i=1}^{n} x_iv_i\right) = \vec{0}$. This means that $\sum_{i=1}^{n} x_iv_i$ is in $N(T)$. Since T is one-to-one, which is equivalent to that $N(T) = \{\vec{0}\}$. So

$$\sum_{i=1}^{n} x_iv_i = \vec{0}.$$

But $\{v_1, \dots, v_n\}$ is linearly independent, this forces $x_1 = \dots x_n = 0$.

(b) It suffices to show that for any $w \in W$, w is a linear combination of $\{T(v_1), \dots, T(v_n)\}$. Since T is onto, that is, $R(T) = W$, then there exists a $v \in V$ such that $T(v) = w$. Since $\{v_1, \dots, v_n\}$ spans V, we have $v = \sum_{i=1}^{n} a_iv_i$ for some $a_i \in F$. Hence

$$w = T(v) = T(\sum_{i=1}^{n} a_iv_i) = \sum_{i=1}^{n} a_iT(v_i).$$

That is, w is a linear combination of $\{T(v_1), \dots, T(v_n)\}$.

(c) T is isomorphism if and only if T is one-to-one and onto. By (a) and (b), we see that $\{T(v_1), \dots, T(v_n)\}$ is linearly independent and spans W. So it is a basis of W.