Math 353, Practice Midterm 2

Name: ______________________________

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	10	
2	12	
3	15	
4	20	
5	20	
6	23	
Total	100	

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

(a) $\det(kA) = k\det(A)$.

(b) If A is a transition matrix then $\lim_{m\to\infty} A^m$ always exists.

(c) If A is diagonalizable then all eigenvalues of A are distinct.

(d) Let $T : V \to V$ be an linear operator and W_1, W_2 two T-invariant subspaces. Then $W_1 \cap W_2$ is also a T-invariant subspace.

(e) Let V be an inner product space with inner product $\langle , \rangle$. If $\langle w, v\rangle = 0$ then either $w = 0$ or $v = 0$.

	(a)	(b)	(c)	(d)	(e)
Answer	F	F	F	T	F

2. Multiple Choice:

(i) Which of the following is NOT equivalent to the statement that A is invertible.

(a) A is diagonalizable.
(b) $\det(A) \neq 0$.
(c) A only has nonzero eigenvalues.
(d) $\text{rank}(A) = n$.
(e) If the characteristic polynomial $f_A(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_t + a_0$ then $a_0 \neq 0$.

The correct answer is (a).

(ii) Suppose $A^3 = A$. Then which of the following is correct.

(a) A is invertible.
(b) $\det(A) = 0$, 1.
(c) Eigenvalues of A are a *subset* of $\{0, \pm 1\}$.
(d) $\lim_{m \to \infty} A^m$ exists.
(e) The dimension of each eigenspace is 1

The correct answer is (C).

(iii) Which of the following properties implies that the $n \times n$ matrix A can be diagonalized?

(a) A is a transition matrix.
(b) A is an invertible matrix.
(c) All eigenvalues of A are same.
(d) The dimension of all eigenspaces is 1.
(e) The algebraic multiplicity of eigenvalue $k_i = 1$ for all i.

The correct answer is (e).

(iv) Consider the following linear system.

$$x + ay + z = b + c$$

$$2x + by + z = a + c$$

$$3x + cy + z = a + b$$

Suppose the system only has unique solution. Then

(a) $x = 0$
(b) $y = 0$
(c) $z = 1$
(d) $x = 1$
(e) $y = 1$

The correct answer is (A).

3. Let

$$A = \begin{pmatrix} 1 & s & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) Find the value of s such that A is diagonalizable.

Solutions: The characteristic polynomial is

$$P_A(\lambda) = \begin{vmatrix} \lambda - 1 & -s & 1 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2)$$

Hence the eigenvalues of A are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. Since the eigenvalue $\lambda_1 = 1$ has multiplicity 2, A is diagonalizable if and only if the dimension of the 1-eigenspace E_1 is 2. Note that the E_1 is given by the solutions of $(1I_3 - A)X = 0$, namely,

$$\begin{pmatrix} 0 & -s & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We easily see that $z = 0$. So if $s \neq 0$ then $y = 0$ and then E_1 is just spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. In this case, the dimension of E_1 is 1, which is less than the multiplicity 2. Hence E_1 has dimension 2 if and only if $s = 0$, in which case, E_1 has a basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(b) For value s that A is diagonalizable, diagonalize A. Namely, find an invertible matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$.

Solutions:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We easily get a basis $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. So we obtain $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

4. Suppose one region has the city and the suburbs. Each year 50% population of the city stay in the city and 50% of them move to the suburbs; 40% population of suburb stays in the suburbs and 60% of them move to the city. Suppose that in 2000, population in the city are 10,000 and population in the suburbs are 20,000.

(a) What is the population in the city and the suburbs in 2002?

Solutions: Let $A = \begin{pmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{pmatrix}$ and $X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ be the vector of population of the city x_n and the suburbs y_n at $2000 + n$ year. It is clear that $X_n = AX_{n-1}$ and hence $X_n = A^n X_0$. So
$X_2 = A^2 X_0 = \begin{pmatrix} 0.55 & 0.54 \\ 0.45 & 0.46 \end{pmatrix} \begin{pmatrix} 10,000 \\ 20,000 \end{pmatrix} = \begin{pmatrix} 16,300 \\ 13,700 \end{pmatrix}$

(b) What is predicted population in the city and suburbs in long run?
Solutions: We need compute that $\lim_{m\to\infty} A^m X_0$. Since A is regular transition matrix. We know $\lim_{m\to\infty} A^m L$ exists and L has identical column v, which is a unique eigenvector with eigenvalue 1. Solve $(A - I)X = 0$, that is

$$\begin{pmatrix} -0.5 & 0.6 \\ 0.5 & -0.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We get $v = k \begin{pmatrix} 6 \\ 5 \end{pmatrix}$. Since v has to be probability vector, $v = \begin{pmatrix} \frac{6}{11} \\ \frac{5}{11} \end{pmatrix}$

Hence $\lim_{m\to\infty} A^m X_0 = 30,000 \times v = \begin{pmatrix} \frac{180,000}{11} \\ \frac{150,000}{11} \end{pmatrix}$.

5. Show that the characteristic polynomial of

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

is

$$f_A(t) = (-1)^n (t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0).$$

Hint: Use cofactor expansion along the first row and then use mathematical induction on n.

proof: We use mathematical induction on n. If $n = 1$ then $f_A(t) = \begin{vmatrix} -t & -a_0 \\ 1 & -t - a_1 \end{vmatrix} = t^2 + a_1 t + a_0$. Suppose $n = k$ the statement is true then for $n = k+1$, we have

$$f_A(t) = \begin{vmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -t - a_k \end{vmatrix}.$$

Using cofactor expansion along the first row, we have

$$f_A(t) = -t \begin{vmatrix} -t & 0 & \cdots & 0 & -a_1 \\ 1 & -t & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -t - a_k \end{vmatrix} + (-1)^{k+2}(-a_0) \begin{vmatrix} 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & -t & \cdots & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & & 1 \end{vmatrix}.$$

By induction on $n = k$, we have

$$\begin{aligned} f_A(t) &= -t\left((-1)^k (t^k + a_k t^{k-1} + \cdots + a_2 t + a_1)\right) + (-1)^{k+1} a_0 \\ &= (-1)^{k+1}(t^{k+1} + a_k t^k + \cdots + a_1 t + a_0). \end{aligned}$$

This proves the case $n = k+1$ and hence complete the induction.

6. Let A be an $n \times n$-matrix.

(a) Show that A is invertible if and only if none of eigenvalues of A is zero.

(b) Suppose A is invertible. Show that if λ is an eigenvalue of A then λ^{-1} is an eigenvalue of A.

(c) Show that A is diagonalizable if and only if A^{-1} is.

proof:

(a) A has an eigenvalue $\lambda = 0$ if and only if there is an eigenvector $v \neq 0$ such that $Av = \lambda v = 0$. So this is equivalent to that the homogeneous equation $AX = 0$ has nontrivial solution, which is equivalent to that A is NOT invertible. So A is invertible if and only if A has no eigenvalue 0.

(b) For the above, we see that $\lambda \neq 0$ and $Av = \lambda v$ with v being eigenvector. Timing A^{-1} on the both side of $Av = \lambda v$, we have $A^{-1}Av = \lambda A^{-1}v$. That is $v = \lambda A^{-1}v$, or equivalently $A^{-1}v = \lambda^{-1}v$. So λ^{-1} is an eigenvalue of A^{-1}.

(c) A is diagonalizable if and only if there exists an diagonal matrix Λ so that A is similar to Λ. Or equivalently there exists an invertible matrix S so that $A = S\Lambda S^{-1}$. If A is invertible then all eigenvalues $\lambda_i \neq 0$. So Λ is invertible because the diagonal of Λ are λ_i. So we have $A^{-1} = (S\Lambda S^{-1})^{-1} = S\Lambda^{-1}S^{-1}$ with Λ^{-1} being diagonal matrix. That is, A^{-1} is also diagonalizable. Since $A = (A^{-1})^{-1}$, A^{-1} is diagonalizable implies that A is diagonalizable.