# Math 353, Midterm 2

Name: ____________________________________________

This exam consists of 8 pages including this front page.

## Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

| *Score* | | |
|---|---|---|
| 1 | 15 | |
| 2 | 20 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 18 | |
| 6 | 17 | |
| *Total* | 100 | |

**Notations:** $\mathbb{R}$ denotes the set of real number and $\mathbb{C}$ denotes the set of complex numbers; $F$ is always a field, for example, $F = \mathbb{R}$; $M_{m\times n}(F)$ denotes the set of $m \times n$-matrices with entries in $F$; $F^n = M_{n\times 1}(F)$ denotes the set of $n$-column vectors; $P_n(F)$ denotes the set of polynomials with coefficients in $F$ and the most degree $n$, that is,

$$P_n(F) = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in F, \ \forall i\}.$$

$V$ is always a finite dimensional vector space over $F$ and $T$ is always a linear operator $T : V \to V$.

**1.** The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. (3 points each)

(a) If $\det(A) = 0$ then rows of $A$ are linearly dependent.

(b) Let $v_1, \ldots, v_m$ be eigenvectors of $A$ with eigenvalues $\lambda_1, \ldots, \lambda_m$. Suppose that $\lambda_1, \ldots, \lambda_m$ are distinct then $v_1, \ldots, v_m$ are linearly independent.

(c) The sum of each row of a transition matrix is 1.

(d) There exists a unique inner product $\langle \ , \ \rangle$ on $\mathbb{C}^n$.

(e) Let $T : V \to V$ be a linear operator. Suppose that $W$ is a $T$-invariant subspace and $\dim_F W = 1$. Then each nonzero vector $v \in W$ is an eigenvector of $T$.

| | (a) | (b) | (c) | (d) | (e) |
|---|---|---|---|---|---|
| Answer | T | T | F | F | T |

**2.** Multiple Choice. (5 points each)

(i) Consider the following linear system.

$$\begin{aligned} x + ay + z &= a + 1 \\ 2x + by + z &= b + 1 \\ 3x + cy + z &= c + 1 \end{aligned}$$

Suppose the system only has unique solution. Then

(a) $x = 1$
(b) $y = 0$
(c) $z = 1$
(d) $x = 2$
(e) $y = 2$

The correct answer is (c).

(ii) Let

$$A = \begin{pmatrix} 0 & 7 & a & 1 \\ 0 & 2 & 0 & 0 \\ 3 & 4 & 5 & 6 \\ 0 & 8 & 9 & a \end{pmatrix}$$

Which of the following statement is correct?

(a) $\det(A) = -6(a^2 - 9)$
(b) $\det(A) = 6(a^2 - 9)$
(c) $\det(A) = 0$.
(d) $A$ is always invertible.
(e) $A$ is invertible if and only if $a \neq 3$.

The correct answer is (a).

(iii) Suppose $T : V \to V$ be a linear operator and $T^3 = T$. Which of the following statement is always correct?

(a) Eigenvalues of $T$ are $\pm 1$.

(b) $T$ must have eigenvalue 0.

(c) Eigenvalues of $T$ are distinct.

(d) Such $T$ is unique.

(e) If $W_x$ is the $T$-cyclic subspace generated by $x \in V$ then $\dim_F W \leq 3$.

The correct answer is (e).

(iv) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Which of the following statement is correct?

(a) $A$ is a transition matrix.

(b) Eigenvalues of $A$ are all distinct.

(c) $A$ is NOT diagonalizable.

(d) All the eigenspaces of $A$ have the same dimension.

(e) $A^3 - 3A^2 = 0$.

The correct answer is (e).

**3.** Let $A = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix}$.

(a) Check that $A$ is a *regular* transition matrix. (7 points)

(b) Does $\lim_{m\to\infty} A^m$ exists? If so calculate $\lim_{m\to\infty} A^m$. (8 points)

*Solutions:*

a) It is easy to see that all entries of $A$ is nonnegative and sum of each column is 1. Recall $A$ is regular if $A^k$ has all positive entries for some $k$. Indeed, $A^2$ has all positive entries. So $A$ is a regular transition matrix.

b) Since $A$ is a regular transition matrix, $L = \lim_{m\to\infty} A^m$ exists. And all columns of $L$ are the same vector $v$ so that $v$ is an eigenvector of $A$ with eigenvalue 1, and $v$ is a probability vector. To find $v$, solve $(A - I)X = 0$ and note that

$$A - I = \begin{pmatrix} -0.5 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \\ 0 & 0.5 & -0.5 \end{pmatrix}$$

We see that $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 1. To make a probability vector, we have $v = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. So

$$\lim_{m\to\infty} A^m = L = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

**4.** Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear operator given by

$$T(f(x)) = f'(x) + 2f(x).$$

(a) Find all eigenvalues $\lambda_i$ of $T$ . (5 points)

(b) For each eigenvalue $\lambda_i$, find a basis of eigenspace

$$E_{\lambda_i} = \{v \in P_2(\mathbb{R}) | T(v) = \lambda_i v\}. \text{ (5 points)}$$

(c) Is $T$ diagonalizable? Why or why not? (5 points)

*Solutions:* a) Take the standard basis $\beta = \{1, x, x^2\}$ of $P_2(\mathbb{R})$, the matrix $A = [T]_\beta$ representing the operator $T$ is determined by

$$T(1, x, x^2) = (0, 1, 2x) = (1, x, x^2) \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

It suffices to find eigenvalues of $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

We easily see the characteristic polynomial of $A$ is $f_A(t) = (2-t)^3$. So eigenvalues of $A$ is 2 with algebraic multiplicity 3.

b) Now $\lambda_i = 2$, to find the eigenspace

$$E_2 = \{v \in P_2(\mathbb{R}) | T(v) = \lambda_i v = 2v\}.$$

We first find the eigenspace $E_2'$ of $A$ for eigenvalue 2. By solving $(A - 2I)X = \vec{0}$, we easily find that $E_2'$ has dimension 1 and spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. So $E_2$ also has dimension 1 and spanned by $f(x) = 1 + 0x + 0x^2 = 1$.

c) Since algebraic multiplicity of $\lambda = 2$ is 3 is larger than the geometric multiplicity $\dim_F E_2 = 1$. So $T$ is NOT diagonalizable.

**5.** Let $A$, $B$ be $n \times n$-matrices. Recall $A$ is *similar* to $B$ if there exists an invertible matrix $S$ so that $A = SBS^{-1}$. Now suppose that $A$ is similar to $B$.

(a) Show that $v$ is an eigenvector of $B$ if and only $Sv$ is an eigenvector of $A$. (6 points)

(b) Show that $A$ and $B$ share the same eigenvalues. More precisely, $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is an eigenvalue of $B$. (6 points)

(c) Show that $A$ is diagonalizable if and only if $B$ is.(6 points)

*Proof*: (a) $v$ is an eigenvector of $B$ if and only if $v \neq \vec{0}$ and $Bv = \lambda v$ with $\lambda$ being eigenvalue. So

$$ASv = (SBS^{-1})Sv = SB(S^{-1}S)v = SBv = S\lambda v = \lambda Sv.$$

Note that $S$ is invertible. So $Sv \neq \vec{0}$. Hence $Sv$ is an eigenvector of $A$ with $\lambda$ the eigenvalue.

Conversely, if $Sv$ is an eigenvalue of $A$ with eigenvalue $\lambda$. That is, $Sv \not\vec{0}$ and $ASv = \lambda Sv$. We have

$$ASv = (SBS^{-1})Sv = SBv = \lambda Sv.$$

Multiplying $S^{-1}$ on the both sides, we get $Bv = \lambda v$. Note that $v \not\vec{0}$ as $S$ is invertible. This shows that $v$ is an eigenvector of $B$ with eigenvalue $\lambda$.

This also proves (b) that $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is an eigenvalue of $A$.

Another proof, we see that characteristic polynomial

$$f_A(t) = |A - tI_n| = |SBS^{-1} - tSI_nS^{-1}| = |S||B - tI_n||S^{-1}| = |B - tI_n| = f_B(t).$$

So $A$ and $B$ share the same characteristic polynomial. Hence $A$ and $B$ share the same eigenvalues.

(c) $B$ is diagonalizable if and only if $B$ has eigenvectors $v_1, \ldots, v_n$ to forms a basis of $F^n$. By (a), $Sv_i$ are eigenvectors of $A$. Since $S$ is invertible, $Sv_1, \ldots, Sv_n$ also forms a basis of $F^n$. Therefore $A$ is diagonalizable. Conversely, if $A$ is diagonalizable, then we can use the same argument the above by replacing $S$ with $S^{-1}$ to show that $B$ is also diagonalizable.

Another proof: $B$ is diagonalizable if and only if there exists an invertible matrix $Q$ so that $B = Q\Lambda Q^{-1}$ with $\Lambda$ a diagonal matrix. Then

$$A = SBS^{-1} = SQ\Lambda S^{-1}Q^{-1} = (SQ)\Lambda(SQ)^{-1}.$$

Since $SQ$ is invertible, $A$ is diagonalizable. Similarly, if $A$ is diagonalizable, that is, $A = Q\Lambda Q^{-1}$. Then $SBS^{-1} = Q\Lambda Q^{-1}$. Then $B = (S^{-1}Q)\Lambda(S^{-1}Q)^{-1}$. Then $B$ is diagonalizable.

**6.** Let $A$ be an $n \times n$-matrix and $\lambda_1, \ldots, \lambda_n$ all its eigenvalues ($\lambda_i$ may not be distinct). Let us show that

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

(a) Show the above statement is true if $A$ is diagonalizable. (6 points)

(b) The proof for the general $A$ is more challenging with following steps:

(i) Show that there exists an invertible matrix $S$ so that $A = SBS^{-1}$ where $B$ has the following shape:

$$B = \begin{pmatrix} \lambda_1 & * \\ \vec{0} & A' \end{pmatrix}$$

where $\vec{0}$ is an $(n-1)$-column zero vector and $A'$ is an $(n-1)\times(n-1)$-square matrix. Hint: Select first column of $S$ to be an eigenvector with eigenvalue $\lambda_1$. (4 points)

(ii) Show that eigenvalues of $A'$ are $\lambda_2, \ldots, \lambda_n$ and $\det(A) = \lambda_1 \det(A')$. (4 points)

(iii) Use mathematical induction on $n$ show that $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$. (3 points)

*Proof:* a) Suppose that $A$ is diagonalizable. Then there exists an invertible matrix $S$ so that $A = S\Lambda S^{-1}$ with $\Lambda$ being a diagonal matrix. Note that $\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$. We have

$$\det(A) = \det(S)\det(\Lambda)\det(S^{-1}) = \det(\Lambda) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

b) i) Let $v_1$ be an eigenvector with eigenvalue $\lambda_1$. Since $v_1 \neq \vec{0}$, one can extend $v_1$ to basis $v_1, v_2, \ldots, v_n$ of $F^n$. Set $S = (v_1, v_2, \ldots, v_n)$. Since

$$AS = (Av_1, Av_2, \ldots, Av_n) = (\lambda_1 v_1, Av_2, \ldots, Av_n) = (v_1, \ldots, v_n)B = SB$$

so that $B$ has the form $B = \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} \end{pmatrix}$. Since columns of $S$ forms a basis, $S$ is invertible. Hence $A = SBS^{-1}$ with $B = \begin{pmatrix} \lambda_1 & * \\ \vec{0} & A' \end{pmatrix}$.

ii) We have $\det(A) = \det(SBS^{-1}) = \det(S)\det(B)\det(S^{-1}) = \det(B)$. Using cofactor expansion on the first column of $B$, we get $\det(A) = \det(B) = \lambda_1 \det(A')$. Since similar matrices share the same characteristic polynomials, we have $f_A(t) = f_B(t)$. Note that

$$f_B(t) = |B - tI_n| = \begin{pmatrix} \lambda_1 - t & * \\ \vec{0} & A' - tI_{n-1} \end{pmatrix}.$$

Using the cofactor expansion on the first column again, we have $f_A(t) = (\lambda_1 - t)|A' - tI_{n-1}| = (\lambda_1 - t) f_{A'}(t)$. So $f_{A'}(t) = 0$ has roots $\lambda_2, \ldots, \lambda_n$. That is, $A'$ has eigenvalues $\lambda_2, \ldots, \lambda_n$.

iii) If $n = 1$, $A = (\lambda_1)$. The statement is clear. Suppose that for $n = k$ the statement is valid. That is, the determinant is the product of all eigenvalues. Now consider $n = k + 1$, from i) and ii), we see that $\det(A) = \lambda_1 \det(A')$ and $A'$ has eigenvalues $\lambda_2, \ldots, \lambda_n$. Since $A$ is a $k \times k$-matrix, by induction, we conclude that $\det(A') = \lambda_2 \cdots \lambda_n$. So

$$\det(A) = \lambda_1 \det(A') = \lambda_1 \lambda_2 \cdots \lambda_n.$$

This completes the induction.