POOL OF FINAL QUESTIONS

(1) What are definitions of number field, algebraic integers, class number?

(2) Definition of Dedekind domain, examples of Dedekind domain nor Dedekind domain.

(3) Give an example $\mathcal{O}_K$ which is not UFD.

(4) What elements of $\mathbb{Q}_p$, $\mathbb{Z}_p$ looks like? Why $m \in \mathbb{Z}$ is an invertible element when $p \nmid m$?

(5) Basic facts about quadratic fields $K = \mathbb{Q}(\sqrt{d})$. Describe $\mathcal{O}_K$, discriminant and decomposition of primes.

(6) Let L/K be a finite Galois extension of number fields. What can say about the relationship of following groups: decomposition group, inertia group, $\mathrm{Gal}(L_w/K_v)$, $\mathrm{Gal}(L/K)$, and $\mathrm{Gal}(l/k)$. Here l is the residue field of L_w and k the residue field of K_v. And how they relate to $e(w|v)$ and $f(w|v)$?

(7) Definition of unramified extension and (totally)ramified extension? can you provide examples? Basic properties of unramified extension: namely composite of unramified extensions are still unramified. Is this true for totally ramified extension? why?

(8) What is Hensel's Lemma (for $\mathbb{Z}_p$)? Give an application of it. E.g., How do we know $\mathbb{Q}_p$ contains more elements than $\mathbb{Q}$?

(9) What is Frobenius? Example of nontrivial Frobenius.

(10) What different, discriminant can tell us about ramifications? In case that $\mathcal{O}_K = \mathbb{Z}[\alpha]$, how should we explicit compute different and discriminant?

(11) Let $E = \mathbb{Q}(\alpha)$ where α is a root of $X^3 + X + 1$. Show that $\mathcal{O}_E = \mathbb{Z}[\alpha]$.

(12) Let $K = \mathbb{Q}(\alpha)$ with $\alpha^3 = 2$. Show that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

(13) Decompose small primes p over the above K with the assumption that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

(14) Recall Minkowski showed that any fractional $\mathcal{O}_K$-ideal $\mathfrak{b}$ there exists an ideal $\mathfrak{a}$ such that $\mathfrak{a} \sim \mathfrak{b}$ and
$$N(\mathfrak{a}) \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{\Delta_K}.$$
Compute the class number for $\mathbb{Q}(\sqrt{D})$ with D small.

(15) Show that $\mathbb{Q}(\zeta_p)$ is totally ramified over p, but unramified over other primes.

(16) Let p be an odd prime. Then there exists a unique quadratic subfield $K \subset \mathbb{Q}(\zeta_p)$, which corresponds to the unique index 2 subgroup for $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Can you explicitly determine K? (Hint, study ramification of K).

(17) State Dirichlet's unit theorem. Compute fundamental unit for $K = \mathbb{Q}(\sqrt{D})$ with small D.

(18) What are (weak) approximation theorem and Chinese remainder theorem? why they are different? Why we couldn't have the following *super* approximation theorem?

Let S be a finite set of valuations containing all $v|\infty$. Given $y_v \in K_v$ for $v \in S$ and $\epsilon > 0$ there exists $x \in K$ so that $|x - y_v|_v < \epsilon, \forall v \in S$ and $|x|_v \leq 1, \forall v \notin S$.