MA 584 HW 1 DUE SEPTEMBER 4TH

(1) Is $\frac{3+2\sqrt{6}}{1-\sqrt{6}}$ an algebraic integer?

(2) Let $K = \mathbb{Q}(\sqrt{5})$.
   (a) Show that $\mathbb{Z}[\sqrt{5}]$ is not integrally closed, it can not be a unique factorization domain.
   (b) Give an example of an element of $\mathbb{Z}[\sqrt{5}]$ that has two distinct factorizations into irreducible elements.

(3) Let $D \in \mathbb{Z}$ be a *square free* integer. Show that

$$\mathcal{O}_K = \{a + b\beta | a, b \in Z\}$$

where

$$\beta = \begin{cases} \sqrt{D} & \text{if } D \equiv 2, 3 \mod 4; \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \mod 4. \end{cases}$$

(4) Let K be a number field, that is, an extension of $\mathbb{Q}$ having degree $n = [K : \mathbb{Q}]$ as a $\mathbb{Q}$-vector space. We say that a subset $\mathcal{O} \subset K$ is an *order* if
   (i) $\mathcal{O}$ is a commutative ring containing 1;
   (ii) $\mathcal{O}$ is a finite generated $\mathbb{Z}$-module;
   (iii) $\mathcal{O}$ contains a $\mathbb{Q}$-basis $w_1, w_2, ..., w_n$ of K, i.e., K is the field of fractions of $\mathcal{O}$.

   Show the following:
   (a) Show that $\mathcal{O} \subset \mathcal{O}_K$.
   (b) Find an example of $\mathcal{O}$ so that $\mathcal{O} \neq \mathcal{O}_K$.

(5) Let $\mathfrak{p}$ be a prime ideal in an integral domain A. Show that $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is the field of fractions of $A/\mathfrak{p}A$.