MA 584 HW 2 DUE SEPTEMBER 18TH

(1) If L/K is a finite separable extension then the bilinear map $L \times L \to K$ induce by $x \times y \to \mathrm{Tr}_{L/K}(xy)$ is nondegenerate. Find an example to show the statement does not hold if L/K is not separable.

(2) Prove Proposition 11 for $\mathfrak{p}$ being a *prime* ideal, instead of maximal ideal assumed in Prop.11. Complete statement: Let A be a ring, integrally closed in its quotient field K. Let L be a finite Galois extension of K with Galois group $G = \mathrm{Gal}(L/K)$. Let $\mathfrak{p}$ be a *prime* ideal of A and let $\mathfrak{P}, \mathfrak{P}'$ be prime ideals of integral closure of$A$ in L lying above $\mathfrak{p}$. Then there exists a $\sigma \in G$ such that $\sigma(\mathfrak{P}') = \mathfrak{P}$.

(3) Let A be a ring, integrally closed in its quotient field K and $x \in L$, a field extension of K. Suppose that x is integral over A. Show that the minimal polynomial $f(X)$ of x has all coefficients in A. Could the assumption that A is integrally closed be dropped?

(4) Let A be a ring and $\mathfrak{a}_i \subset A$, $i = 1, \ldots, n$ be (proper) ideals. Assume that $\mathfrak{a}_i, \mathfrak{a}_j$ are relatively prime, i.e., $\mathfrak{a}_i + \mathfrak{a}_j = A, \forall i \neq j$.
   (a) Show that there exists an isomorphism

$$A/\mathfrak{c} \simeq A/\mathfrak{a}_1 \times \cdots \times A/a_n.$$

   (b) Let k be a field and A a finite generated k-algebra. Assume that A is a finite k-vector space and A is reduced. Show that there exist some field $K_1, \ldots, K_n$ so that $A \simeq K_1 \times \cdots \times K_m$

(5) Let k be a field, $A = k[X]$, $K = k(X)$ and L is a finite separable extension of K. Let B be a integral closure of A in L. Show that B is a Dedekind domain. This is the case of function field, parallel to the case of number field. Is $k[X, Y]$ a Dedekind Domain?

(6) Let $F = \mathbb{F}_q$ be a finite field with $q = p^n$-elements. Show the following
   (a) For each $n \in \mathbb{N}$ there exists a unique field extension L_n/F so that $[L_n : F] = n$.

(b) L_n/F is Galois and $\mathrm{Gal}(L_n/F)$ is cyclic with a canonical generator (Frobenius) defined by $x \to x^q, \forall x \in L_n$.