MA 584 HW 5 DUE OCTOBER 30TH

Let $(A, \mathfrak{p})$ be a complete DVR with uniformizer π, that is, $\mathfrak{p} = (\pi)$. Write $k := A/\mathfrak{p}$ the residue field.

(1) Prove an Eisenstein Polynomial is always irreducible. Recall a polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is Eisenstein if $\pi | a_0, \ldots, a_{n-1}$ and $\pi^2 \nmid a_0$.

(2) We have filtration $A \supset \mathfrak{p} \supset \mathfrak{p}^2 \supset \cdots \mathfrak{p}^n \supset \cdots$ and

$$A^\times \supset 1 + \mathfrak{p} \supset 1 + \mathfrak{p}^2 \supset \cdots 1 + \mathfrak{p}^n \supset \cdots$$

(a) Show that $1+\mathfrak{p}^n$ is a subgroup of $A^\times$. Indeed, $1+\mathfrak{p}^n$ forms a systems of open neighborhoods of 1.
(b) Show that $\mathfrak{p}^n/\mathfrak{p}^{n+1} \simeq k$ as additive groups.
(c) $A/1+\mathfrak{p} \simeq k^\times$ and $1+\mathfrak{p}^n/1+\mathfrak{p}^{n+1} \simeq k$ for $n \geq 1$ as groups.

(3) Let K be a finite extension of $\mathbb{Q}_p$. Fix an integer $n > 0$. Consider the set

$$S_n = \{L | L \text{ is a finite extension of } K \text{ so that } [L : K] = n\}.$$

Show that S_n is a finite set.

Hint: Use the fact that A is compact and Problem (2) above and the following result: Given $f \in K[X]$ then there exists an $\epsilon > 0$ depending on f so that if $|f - g| < \epsilon$ and α is a root of $f(x)$ then there exists a root β of $g(x)$ such that $K(\alpha) = K(\beta)$.

(4) Let $\Phi_m(x) = x^m - 1$. Let ζ_m be a root of $\Phi_m(x)$ and there does not exists n so that $0 < n < m$ and $\zeta_m^n = 1$.
(a) If $m|p-1$ show that $\zeta_m \in \mathbb{Q}_p$ (note in general, $\zeta_m \notin \mathbb{Q}$).
(b) If $p \nmid m$ show that $\mathbb{Q}_p(\zeta_m)$ is unramified over $\mathbb{Q}_p$.
(c) (Extra credits) Show that any unramified extension K of $\mathbb{Q}_p$ is contained in $\mathbb{Q}_p(\zeta_m)$ for an m with $p \nmid m$.

(5) The composite of two unramified extensions are still unramified. Is this still true if we replace *unramified* by *totally ramified*?

(6) Let L/K be finite separable extension and assume that k is perfect with characteristic p. We have shown that there exists field extensions

$$K \subset L_0 \subset L_t \subset L$$

so that L_0/K is maximal unramified, i.e., L_0 is unramified over K and $f(L/K) = f(L_0/K)$; L_t/L_0 is maximal tamely totally ramified, i.e., if $e(L/K) = p^s m$ with $p \nmid m$ then L_t/L_0 is totally ramified and $[L_t : L_0] = m$. Here we show that L_0 and L_t can be constructed as the following:

Pick E/K a Galois closure of L and write $G := \mathrm{Gal}(E/K)$. Let k and k_E be residue field of K and E respectively. Then we have a natural projection $G \twoheadrightarrow \mathrm{Gal}(k_E/k)$ by Chapter 1 Prop 14. Recall the kernel of this map is called the inertia subgroup I. If H is a subgroup of G then recall

$$E^H = \{x \in E | g(x) = x, \forall g \in H\}$$

(a) Show that E^I is the maximal unramified subextension of E.
(b) Let I_p be the p-sylow subgroup of G. Show that E^{I_p}/E^I is the maximal totally tamely ramified extension.
(c) Show that $L_0 = L \cap E^I$ and $L_t = L \cap E^{I_p}$.