MA 584 HW 7 DUE NOV. 30TH

In the following, K is always a number field. Recall Minkowski showed that any fractional $\mathcal{O}_K$-ideal $\mathfrak{b}$ there exists an ideal $\mathfrak{a}$ such that $\mathfrak{a} \sim \mathfrak{b}$ and

$$N(\mathfrak{a}) \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{\Delta_K}.$$

(1) Show that the following quadratic field $\mathbb{Q}(\sqrt{D})$ has class number 1. $D = 5, -3, 2, -7$.

(2) Show that Δ_K goes to infinity when $n = [K : \mathbb{Q}]$ goes to infinity.

(3) (a) Let $\mathfrak{a} \subset \mathcal{O}_K$ be an ideal. Suppose that $\mathfrak{a}^m = a\mathcal{O}_K$ for $a \in \mathcal{O}_K$. Show that $\mathfrak{a}$ become principal in the field $L = K(\sqrt[m]{a})$.
(b) Show there exists a finite extension L of K so that all ideal of K become principal in L.

(4) Let ζ_m be primitive m-th root of unity. Show that $\frac{1-\zeta^k}{1-\zeta}$ for $(k, m) = 1$ are units of $\mathcal{O}_{\mathbb{Q}(\zeta_m)}$. These units are called *cyclotomic units.*

(5) We have shown that if $p \nmid m$ then $\mathbb{Q}(\zeta_m)$ is unramified over p.
(a) Explicitly determine Frobenius at p. Namely, let χ denote the isomorphism $\chi : \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$. Then what is the image of Frobenius at p under χ?
(b) Show that p splits in $\mathbb{Q}(\zeta_m)$ if and only if $p \equiv 1 \mod m$.

(6) Let p be an odd prime. Then there exists a unique quadratic subfield $K \subset \mathbb{Q}(\zeta_p)$, which corresponds to the unique index 2 subgroup for $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Can you explicitly determine K? (Hint, study ramification of K).