Section 3.2.

Taylor's theorem

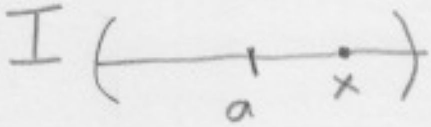
For functions of 1-variable $f: \mathbb{R} \to \mathbb{R}$, Taylor's Theorem says that if $f(x)$ has $k+1$ continuous derivatives in an open interval I centered at $x=a$, then, for all $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$+ \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x),$$

where the error $R_k(x)$ is given by:

$$R_k(x) = \int_a^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\, dt$$



Ex: $f(x) = \ln x, \quad a = 1$

$f(1) = 0$

$f'(x) = \frac{1}{x}, \quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}, \quad f'''(1) = 2$

$\vdots$

$f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{x^k}, \quad f^{(k)}(1) = (-1)^{k-1}(k-1)!$

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots + \frac{(-1)^{k-1}}{k}(x-1)^k + R_k(x)$$

where

$$R_k(x) = \int_1^x \frac{(x-t)^k}{k!} \frac{(-1)^k k!}{t^{k+1}}\, dt = (-1)^k \int_1^x \frac{(x-t)^k}{t^{k+1}}\, dt$$

Taylor's theorem for functions $f: \mathbb{R}^2 \to \mathbb{R}$.

Suppose $f(x,y)$ has continuous derivatives up to an including order $k+1$ in a neighborhood of (x_0, y_0). Then, for (x,y) close enough to (x_0,y_0):

$$\begin{aligned} f(x,y) = {} & f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ & + \frac{1}{2}\Big(f_{xx}(x_0,y_0)(x-x_0)^2 + 2 f_{xy}(x_0,y_0)(x-x_0)(y-y_0) \\ & \qquad + f_{yy}(x_0,y_0)(y-y_0)^2 \Big) \\ & + \frac{1}{3!}\Big(f_{xxx}(x_0,y_0)(x-x_0)^3 + 3 f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0) \\ & \qquad + 3 f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2 + f_{yyy}(x_0,y_0)(y-y_0)^3 \Big) \\ & + \dots + \frac{1}{k!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y} \right)^k f + R_k(x,y) \end{aligned}$$

where $R_k(x,y) \to 0$ as $(x,y) \to (x_0,y_0)$ faster than any of the other terms.

Recall that if $z = f(x,y)$ (Graph case), then the equation of the tangent plane at $(x_0, y_0, f(x_0,y_0))$ is given by:

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0),$$

and this is the first order approximation of f; that is, the Taylor's formula is stopped at $k=1$:

$$f(x,y) \cong f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

with an error $R_1(x,y)$ that goes to zero as $(x,y) \to (x_0,y_0)$.

The second order approximation is:

$$f(x,y) \cong f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$
$$+\frac{1}{2}\Big(f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0)$$
$$+ f_{yy}(x_0,y_0)(y-y_0)^2\Big)$$

with an error $R_2(x,y)$ that goes to zero as $(x,y) \to (x_0,y_0)$.

Ex: Find the second order approximation to $f(x,y) = e^{-x^2-y^2}\cos(xy)$ at $x_0 = 0$, $y_0 = 0$

We use the previous formula:

$f(0,0) = 1$

$\frac{\partial f}{\partial x} = -2x\, e^{-x^2-y^2}\cos(xy) - y\, e^{-x^2-y^2}\sin(xy)$ $\quad \frac{\partial f}{\partial x}(0,0) = 0$

$\frac{\partial f}{\partial y} = -2y\, e^{-x^2-y^2}\cos(xy) - x\, e^{-x^2-y^2}\sin(xy)$ $\quad \frac{\partial f}{\partial y}(0,0) = 0$

$f_{xx} = -2\, e^{-x^2-y^2}\cos(xy)$ + terms which will be 0 at $(0,0)$

$f_{xx}(0,0) = -2$

Similarly, $f_{yy}(0,0) = -2$

$f_{xy} = f_{yx} =$ terms which will be 0 at $(0,0)$

$f_{xy}(0,0) = 0$.

Second order approximation is:

$$f(x,y) \cong 1 + \tfrac{1}{2}(-2x^2 - 2y^2)$$
$$= 1 - x^2 - y^2$$