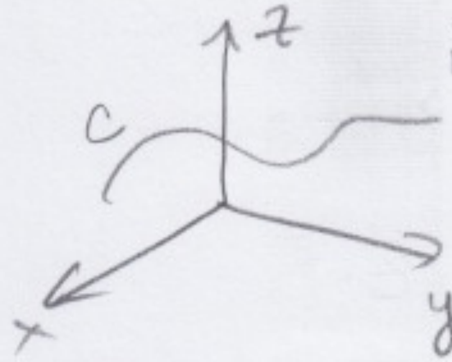
Section 7.3

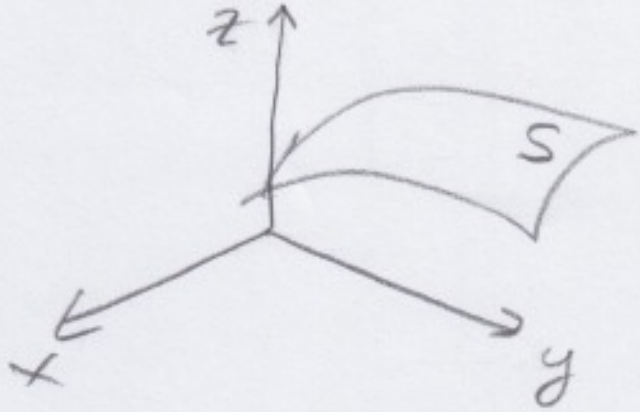


$\vec{r}(t) = (x(t), y(t), z(t))$

$f: \mathbb{R}^3 \to \mathbb{R}$

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$

$\int_C f\,ds \qquad \int_C \vec{F} \cdot d\vec{r}$

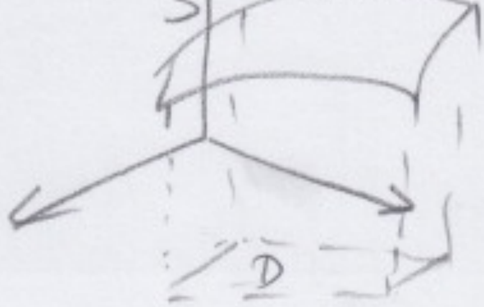


$\iint_S f\,dS$

$\iint_S \vec{F} \cdot d\vec{S}$

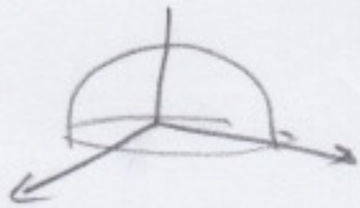
3 common ways to describe a surface

(1) Function description



$z = f(x, y)$

Ex: $z = \sqrt{4 - x^2 - y^2}$



(2) Level surface. (equation description)

$$F(x, y, z) = c$$

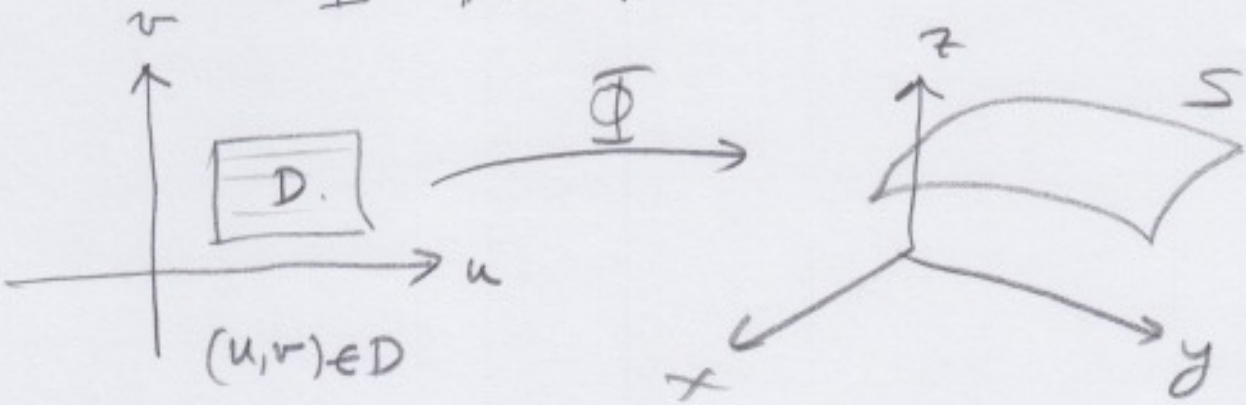
Ex: $F(x, y, z) = x^2 + y^2 + z^2$

$$x^2 + y^2 + z^2 = 4$$

(3) Parametric description

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^3$$



If Φ is C^1 then S is called a C^1 surface.

Ex: Consider the cone.

This cone can be given by a function description

$$z = f(x,y) = \sqrt{x^2+y^2}$$

It can also be given by a parametric description

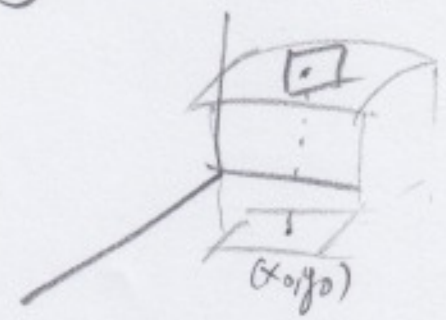
$$\Phi(u,v) = (\underbrace{u\cos v}_{x}, \underbrace{u\sin v}_{y}, \underbrace{u}_{z}), \quad u \geqslant 0$$

We can check out that the parametric description gives the cone by substitution in to $z = \sqrt{x^2+y^2}$

$$u = \sqrt{u^2\cos^2 v + u^2\sin^2 v}$$

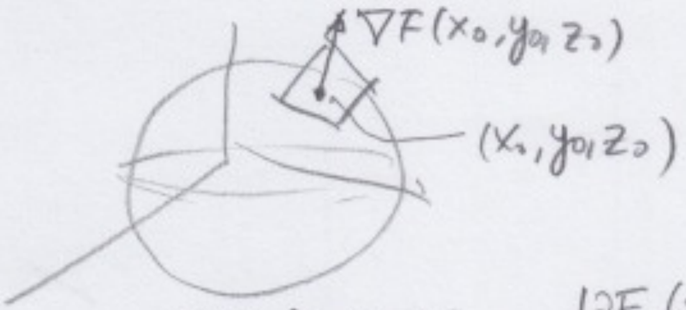
$$= \sqrt{u^2} = u \quad \checkmark$$

1. $z = f(x,y)$. $\quad f: \mathbb{R}^2 \to \mathbb{R}$



$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

2. $F(x,y,z) = c$ $\quad F: \mathbb{R}^3 \to \mathbb{R}$

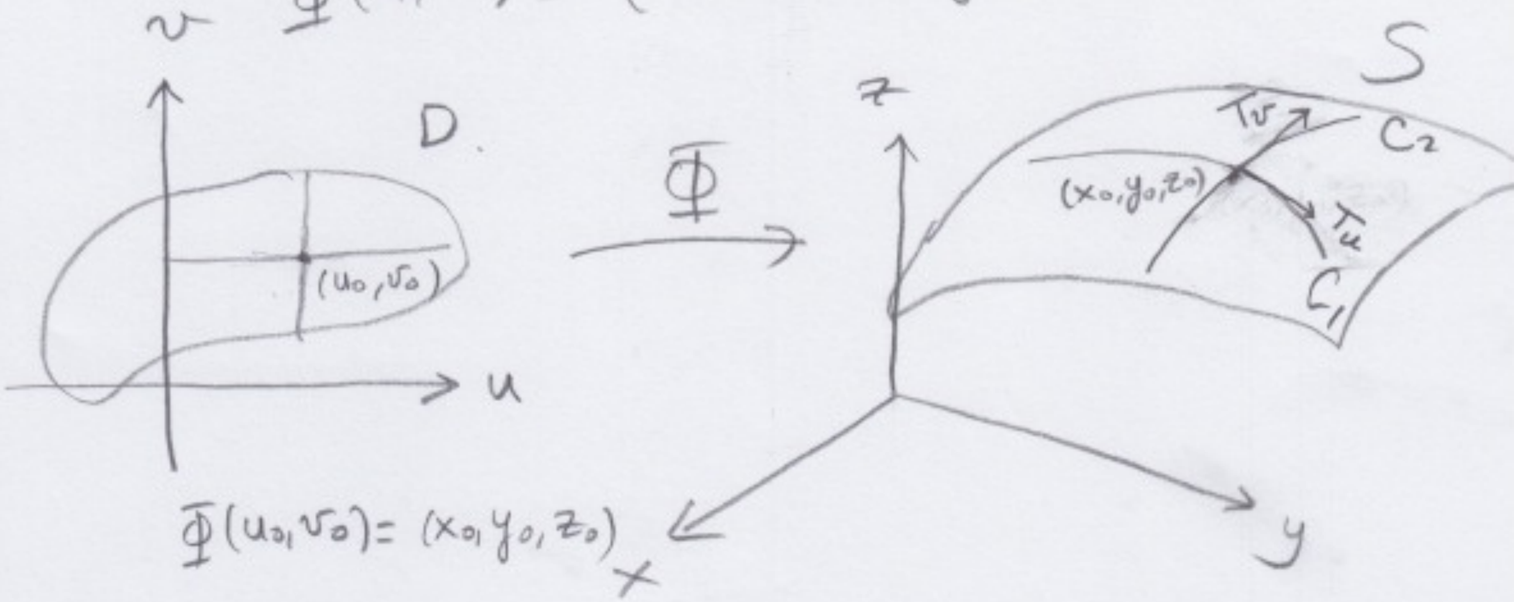


$$\nabla F(x_0,y_0,z_0) = \left(\frac{\partial F}{\partial x}(x_0,y_0,z_0), \frac{\partial F}{\partial y}(x_0,y_0,z_0), \frac{\partial F}{\partial z}(x_0,y_0,z_0)\right)$$

$$\frac{\partial F}{\partial x}(x_0,y_0,z_0)(x-x_0) + \frac{\partial F}{\partial y}(x_0,y_0,z_0)(y-y_0) + \frac{\partial F}{\partial z}(x_0,y_0,z_0)(z-z_0) = 0$$

3. Need to find the equation for the tangent plane when S is given by parametrization

$\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$



$\Phi(u_0,v_0) = (x_0,y_0,z_0)$

C_1: $x = x(u,v_0)$, $y = y(u,v_0)$, $z = z(u,v_0)$. $u_0 - \varepsilon < u < u_0 + \varepsilon$

C_2: $x = x(u_0,v)$, $y = y(u_0,v)$, $z = z(u_0,v)$, $v_0 - \varepsilon < v < v_0 + \varepsilon$

$$\vec{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \quad \vec{T}_u(u_0,v_0)$$

$$\vec{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right). \quad \vec{T}_v(u_0,v_0)$$

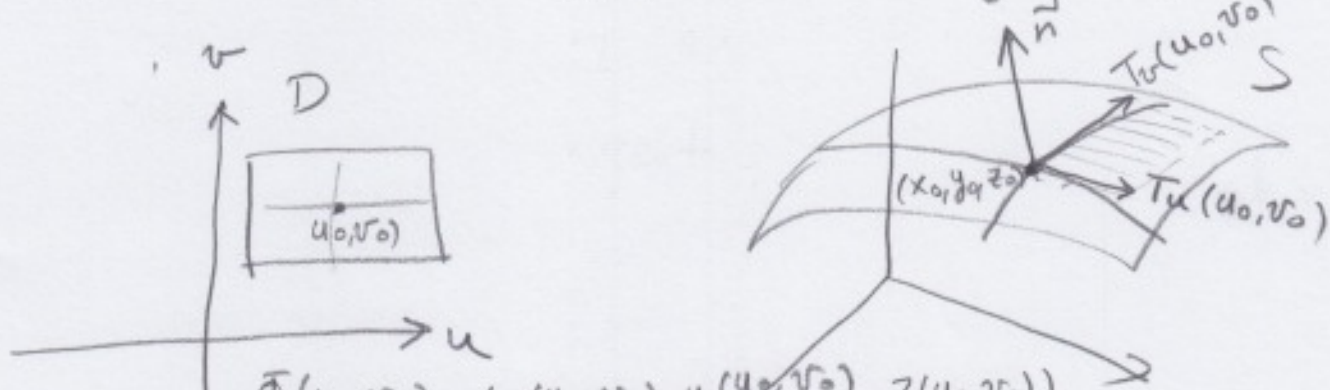
$\vec{T}_u(u_0,v_0)$ is the tangent vector to C_1 at (x_0,y_0,z_0)

$\vec{T}_v(u_0,v_0)$ is the tangent vector to C_2 at (x_0,y_0,z_0)

Since $\vec{T}_u(u_0,v_0)$ and $\vec{T}_v(u_0,v_0)$

are in the tangent plane,

$\vec{n} = \vec{T}_u(u_0,v_0) \times \vec{T}_v(u_0,v_0)$ gives us a normal vector for the tangent plane.



$\Phi(u_0,v_0) = (x(u_0,v_0), y(u_0,v_0), z(u_0,v_0))$

Ex. If S is given by

$\Phi(u,v) = (u^2-v^2, u+v, u^2+4v)$, find the tangent plane at $(-\frac{1}{4}, \frac{1}{2}, 2)$.

$$u^2 - v^2 = -\frac{1}{4} \Rightarrow (u-v)(\underline{u+v}) = -\frac{1}{4}$$
$$u+v = \frac{1}{2}$$
$$u^2+4v = 2 \checkmark \quad \begin{cases} u-v = -\frac{1}{2} \\ u+v = \frac{1}{2} \end{cases} \Rightarrow \begin{matrix} 2u = 0 \\ u = 0 \\ v = \frac{1}{2} \end{matrix}$$

$$(u_0, v_0) = (0, \tfrac{1}{2})$$
$$\vec{T}_u = (2u, 1, 2u), \quad \vec{T}_v = (-2v, 1, 4)$$
$$\vec{T}_u(0, \tfrac{1}{2}) = (0, 1, 0) \quad \vec{T}_v(0, \tfrac{1}{2}) = (-1, 1, 4)$$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -1 & 1 & 4 \end{vmatrix}$$

$$\vec{n} = \vec{i}\,(4-0) - \vec{j}\,(0-0) + \vec{k}\,(1)$$

$$\vec{n} = (4, 0, 1) \qquad (x_0, y_0, z_0) = \left(-\tfrac{1}{4}, \tfrac{1}{2}, 2\right)$$

$$4\left(x + \tfrac{1}{4}\right) + 0\left(y - \tfrac{1}{2}\right) + 1 \cdot (z - 2) = 0$$

$$4x + 1 + z - 2 = 0$$

$$\boxed{4x + z = 1}$$

Ex. For the cone $z = \sqrt{x^2 + y^2}$, find the equation of the tangent plane at $(1, 1, \sqrt{2})$.

Sol. 1: $\Phi(u, v) = (u\cos v, u\sin v, u)$

$u_0 \cos v_0 = 1$

$u_0 \sin v_0 = 1$

$u_0 = \sqrt{2}$

$v_0 = \frac{\pi}{4}$

$u_0 = \sqrt{2}$

$(1, 1, \sqrt{2})$

$\vec{T_u} = (\cos v, \sin v, 1) \quad \vec{T_v} = (-u\sin v, u\cos v, 0)$

$\vec{T_u}\left(\sqrt{2}, \frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right) \quad \vec{T_v}\left(\sqrt{2}, \frac{\pi}{4}\right) = (-1, 1, 0)$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$\vec{n} = \vec{i}\,(-1) - \vec{j}\,(1) + \vec{k}\,\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$\vec{n} = \left(-1, -1, \frac{2}{\sqrt{2}}\right) = (-1, -1, \sqrt{2})$$

$$(x_0, y_0, z_0) = (1, 1, \sqrt{2})$$

$$-1\,(x-1) - 1\,(y-1) + \sqrt{2}\,(z - \sqrt{2}) = 0.$$

$$\frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1) = z - \sqrt{2}$$

or $\quad z = \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

Solution 2:

$z = \sqrt{x^2+y^2} = f(x,y) \qquad \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

$$\frac{\partial f}{\partial x}(1,1) = \frac{1}{\sqrt{2}} \qquad \frac{\partial f}{\partial z}(1,1) = \frac{1}{\sqrt{2}}$$

$$f(1,1) = \sqrt{2}$$

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\,(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$z = \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$$