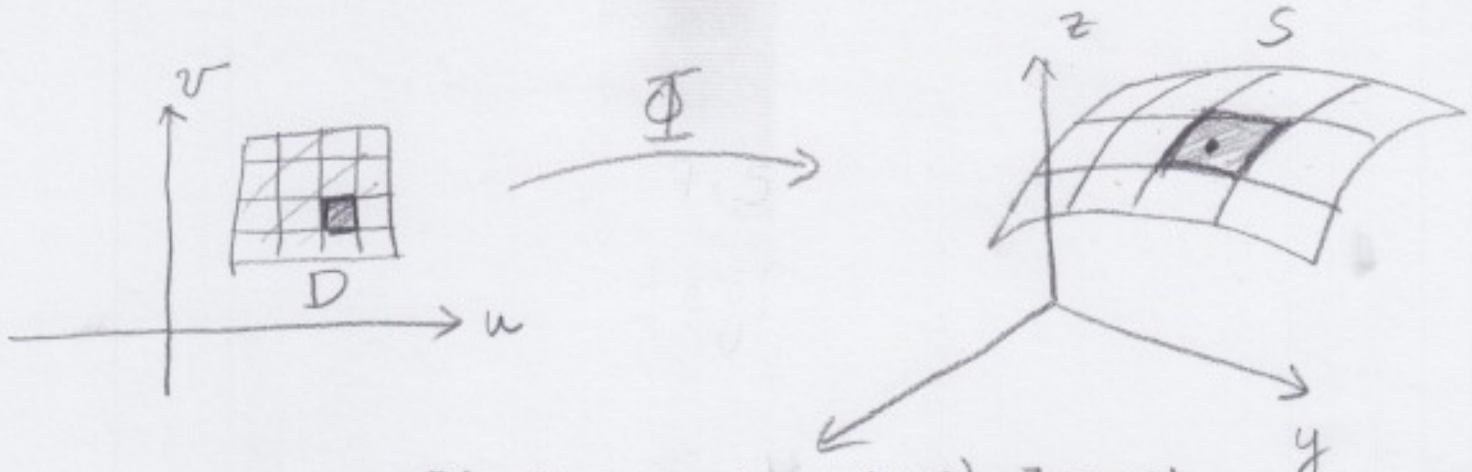
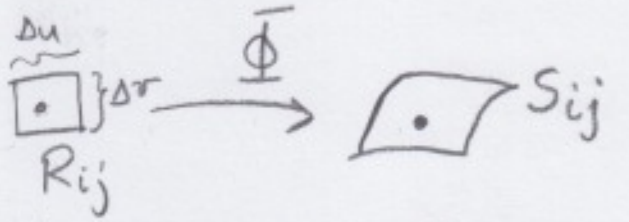
Section 7.5
Surface integrals.



$$\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

Let $f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$. Assume f is the density (gr/unit area) of S.

Question: Compute mass of S.



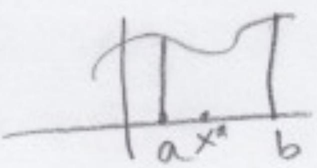
$\Phi(R_{ij}) = S_{ij}$

From Section 7.4 :

$$A(S_{ij}) = \iint_{R_{ij}} \|\vec{T}_u \times \vec{T}_v\|\, du\, dv. \quad \exists (u_i^*, v_j^*)$$

$$\underset{\text{Theorem}}{\overset{\text{Mean Value}}{=}} \|\vec{T}_u(u_i^*, v_j^*) \times \vec{T}_v(u_i^*, v_j^*)\| \underline{\Delta u \Delta v}$$

Recall:



$$\int_a^b f(x)\,dx = f(x^*)\underline{(b-a)}$$

$$M(S_{ij}) \cong f(\Phi(u_i^*, v_j^*))\, \|\vec{T}_u(u_i^*, v_j^*) \times \vec{T}_v(u_i^*, v_j^*)\| \Delta$$

$$M(S) = \sum_{i=1}^{n} \sum_{j=1}^{m} M(S_{ij}) \quad \left(\frac{gr}{\cancel{\text{unit area}}} \right) \cancel{(\text{area})} = gr$$

$$= \lim_{\Delta u, \Delta v \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\Phi(u_i^*, v_j^*)) \, \| \vec{T}_u (u_i^*, v_j^*) \times \vec{T}_v (u_i^*, v_j^*) \| \, \Delta u \Delta v$$

$$= \iint_D f(\Phi(u,v)) \, \| \vec{T}_u \times \vec{T}_v \| \, du \, dv$$

$$= \iint_D f\big(x(u,v), y(u,v), z(u,v)\big) \, \| \vec{T}_u \times \vec{T}_v \| \, du \, dv$$

If $f \equiv 1$, Then we are computing the area of S.

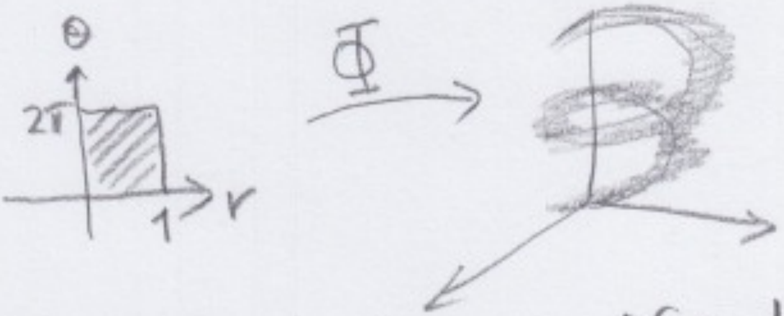
<u>Notation</u>: $\displaystyle\iint_S f \, dS$

Ex. Let $f(x,y,z) = \sqrt{1+x^2+y^2}$

and S be the helicoid

given by:

$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta$$
$$z(r,\theta) = \theta \quad , \; 0 \le r \le 1 \; , \; 0 \le \theta \le 2\pi$$



Compute $\iint_S f \, dS$.

$$\iint_S f \, dS = \iint_D f(\Phi(r,\theta)) \, \|\vec{T}_r \times \vec{T}_\theta\| \, dr \, d\theta$$

$$\vec{T}_r = (\cos\theta, \sin\theta, 0)$$
$$\vec{T}_\theta = (-r\cos\theta, r\sin\theta, 1)$$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\cos\theta & r\sin\theta & 1 \end{vmatrix} = \vec{i}(\sin\theta) - \vec{j}(\cos\theta) + \vec{k}(r)$$

$$= \int_0^{2\pi} \int_0^1 \sqrt{1+r^2} \, \sqrt{1+r^2} \, dr \, d\theta$$
$$= 2\pi \int_0^1 (1+r^2) \, dr$$
$$= 2\pi \left[r + \frac{r^3}{3} \right]_0^1 = 2\pi \cdot \left(1 + \frac{1}{3}\right) = \frac{8\pi}{3}.$$

Section 7.5 continuation

Recall from last video that if a surface S is given as the graph $f(x,y)$, $(x,y)\in D$, then by forming the parametrization $\Phi(x,y)=(x,y,f(x,y))$, $(x,y)\in D$, we obtained:

$$A(S)=\iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$$

Moreover,

$$\iint_S g\,dS=\iint_D g(x,y,f(x,y))\underbrace{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}_{\|\vec{T}_x\times\vec{T}_y\|}\,dx\,dy$$

Ex: Let S be given by $f(x,y)=x^2+y$. Evaluate $\iint_S \frac{xyz}{x^2+y}\,dS$. $(x,y)\in D$, $D=[0,1]\times[0,1]$

$$\iint_S \frac{xyz}{x^2+y}\,dS=\int_0^1\int_0^1 \frac{xy\,\cancel{(x^2+y)}}{\cancel{x^2+y}}\sqrt{1+(2x)^2+(1)^2}\,dx\,dy$$

$$=\int_0^1\int_0^1 xy\sqrt{2+4x^2}\,dx\,dy$$

$$=\int_0^1 \frac{y}{8}\left[\frac{(2+4x^2)^{3/2}}{3/2}\right]_0^1 dy.$$

$$=\frac{1}{12}\int_0^1 y\,(6^{3/2}-2^{3/2})\,dy=\frac{1}{24}(6^{3/2}-2^{3/2}).$$

Ex: Consider the helicoid

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = \theta$$

$$0 \le \theta \le 2\pi,\quad 0 \le r \le 1 \qquad D = [0,2\pi]\times[0,1]$$

has mass density twice the distance from z axis. Find total mass.

$$\rho(x,y,z) = 2\sqrt{x^2+y^2}$$

$$M(S) = \iint_S \rho(x,y,z)\,dS = \iint_S 2\sqrt{x^2+y^2}\,dS$$

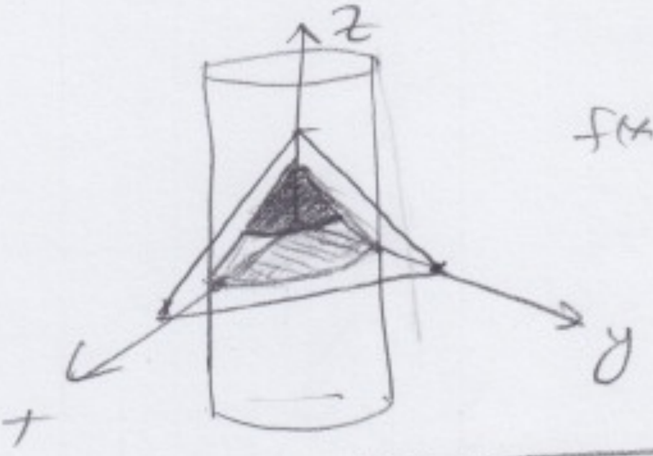
$$\|\vec{T}_r \times \vec{T}_\theta\| = \sqrt{1+r^2}$$

$$= \int_0^{2\pi}\int_0^1 2\sqrt{r^2}\cdot\sqrt{1+r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r\sqrt{1+r^2}\,dr\,d\theta = \frac{4}{3}\pi\left(2^{3/2}-1\right)$$

Note: We are not making a change of variable to polar coordinates.

Ex: Find the area of the surface defined by $x+y+z=2$, $x^2+y^2 \le 1$ in the first octant.



$f(x,y) = z = 2-x-y$

$(x,y) \in D$.

D is circle $x^2+y^2 \le 1$ in first octant.

$$A(S) = \iint\limits_{\substack{x^2+y^2\le 1 \\ \text{(first octant)}}} \sqrt{1+(-1)^2+(-1)^2}\, dx\,dy$$

$$= \sqrt{3} \iint\limits_{\substack{x^2+y^2\le 1 \\ \text{(first octant)}}} dx\,dy$$

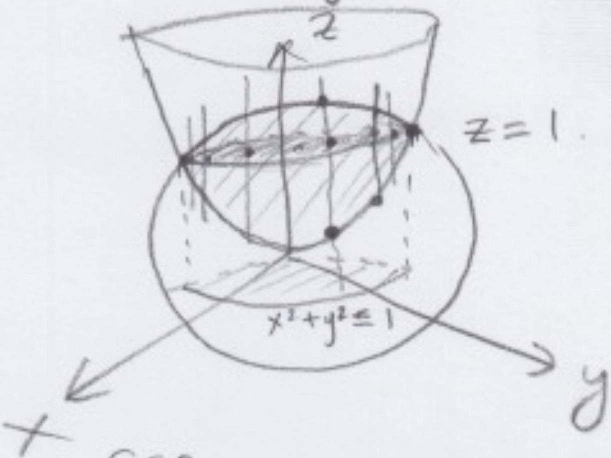
polar coordinates

$$= \sqrt{3} \int_0^{\pi/2}\int_0^1 r\, dr\, d\theta = \sqrt{3}\,\frac{\pi}{2}\cdot\frac{1}{2}$$

$$= \sqrt{3}\,(\pi/4).$$

Quiz

1. Compute the volume inside $z = x^2+y^2$ and $x^2+y^2+z^2 = 2$.



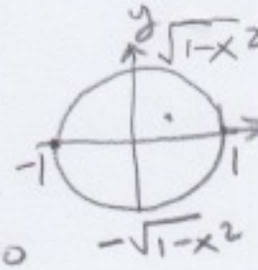
$x^2+y^2+z^2=2$

$z+z^2=2$

$z^2+z-2=0.$

$(z+2)(z-1)=0$

$z=1.$



$$\iiint_W 1\, dx\,dy\,dz = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} 1\, dz\,dy\,dx$$

Use cylindrical coordinates.

$$= \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} \cdot \underset{\uparrow \text{Jacobian}}{r}\, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\left[\sqrt{2-r^2} - r^2\right] dr\,d\theta$$

$$= (2\pi)\left[\int_0^1 r\sqrt{2-r^2}\,dr - \int_0^1 r^3\,dr\right]$$

$$= (2\pi)\left[-\frac{1}{2}\left[\frac{(2-r^2)^{3/2}}{3/2}\right]_0^1 - \left[\frac{r^4}{4}\right]_0^1\right]$$

$$= 2\pi\left[-\frac{1}{3}\left(1-2^{3/2}\right) - \frac{1}{4}\right] = -\frac{2\pi}{3} + \frac{2\pi}{3}2^{3/2} - \frac{\pi}{2}.$$