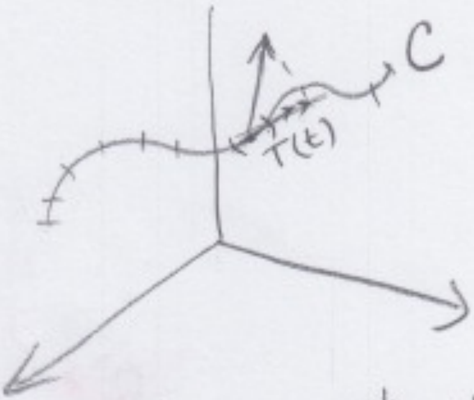
Section 7.6

Surface integrals of vector fields.

Recall:

$f: \mathbb{R}^3 \to \mathbb{R}$ scalar function or real valued function

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ vector field.

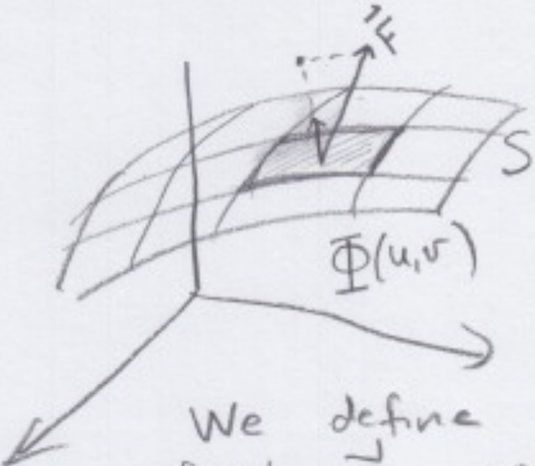


$C \quad \vec{r}(t) = (x(t), y(t), z(t)), \quad a \le t \le b$

$$\int_C f \, \underline{\underline{ds}} = \int_a^b f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt.$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \underbrace{\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}_{\vec{T}(t)} \|\vec{r}\,'(t)\| \, dt.$$

$$= \int_C \underbrace{\vec{F} \cdot \vec{T}} \, ds$$



$$\iint_S f \, \underline{\underline{dS}} = \iint_D f(\Phi(u,v)) \, \|\vec{T}_u \times \vec{T}_v\| \, du$$

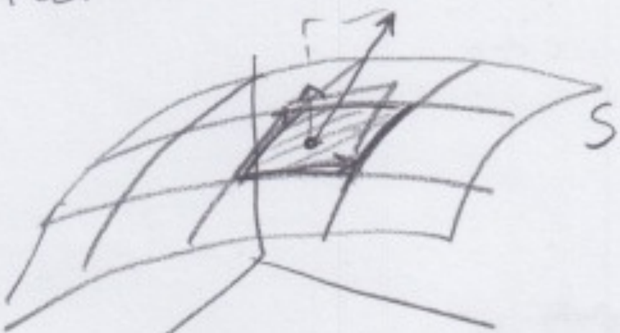
We define the integral of a vector field $\vec{F}$ on S as:

$$\iint_S \vec{F} \cdot d\vec{S} := \iint_S \underbrace{\vec{F} \cdot \vec{n}}_{m/sec} \, \underbrace{dS}_{m^2} = m^3/sec$$

Physical interpretation of vector surface integrals

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS$ is the net quantity of fluid to flow across the surface per unit time; that is, the rate of fluid flow.

$\iint_S \vec{F} \cdot \vec{n} \, dS$ is called "the flux of $\vec{F}$ across the surface".



$$\lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} (\vec{F} \cdot \vec{n})(\Phi(u_i^*, v_j^*)) \|\vec{T}_u(u_i^*, v_j^*) \times \vec{T}_v(u_i^*, v_j^*)\| \Delta u \Delta v$$

$$m/\text{sec} \cdot m^2 = m^3/\text{sec}$$

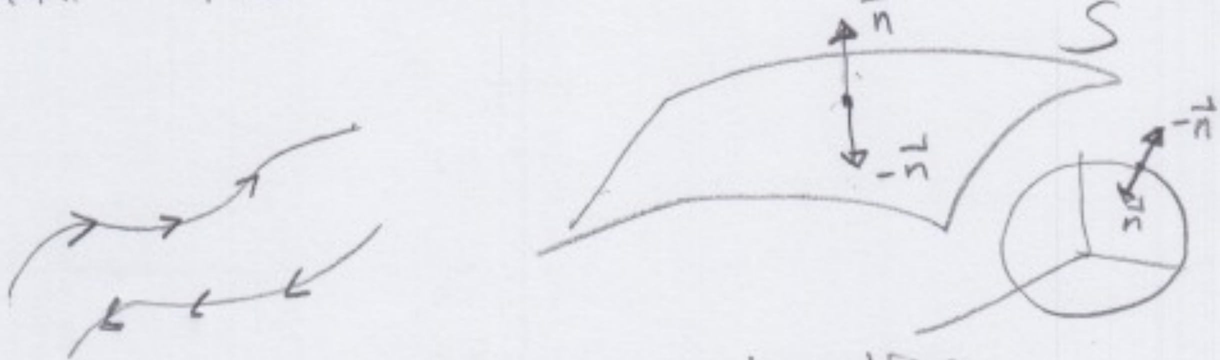
$$= \iint_D (\vec{F} \cdot \vec{n})(\Phi(u,v)) \|\vec{T}_u(u,v) \times \vec{T}_v(u,v)\| \, du \, dv$$

$$= \iint_S \vec{F} \cdot \vec{n} \, dS.$$

Recall that $\int_C f\,ds$ does not depend on the parametrization $\vec{r}(t)$ of C,

and that $\int_C \vec{F}\cdot d\vec{r}$ depends only on the orientation of C (if we reverse the orientation, we change the sign).

For $\iint_S \vec{F}\cdot\vec{n}\,dS$ we must also deal with the orientation of S.



Remark: Given a parametrization $\Phi(u,v)$, $\dfrac{\vec{T}_u\times\vec{T}_v}{\|\vec{T}_u\times\vec{T}_v\|}$ is a vector perperdicular to the surface. It could be $\vec{n}$ or it could be $-\vec{n}$

$$\iint_S \vec{F}\cdot\vec{n}\,\underline{\underline{dS}} = \iint_D \vec{F}(\Phi(u,v))\cdot\frac{\vec{T}_u\times\vec{T}_v}{\cancel{\|\vec{T}_u\times\vec{T}_v\|}}\ \cancel{\|\vec{T}_u\times\vec{T}_v\|}\,du\,dv$$

change sign if necessary (ie. $\dfrac{\vec{T}_v\times\vec{T}_u}{\|\vec{T}_v\times\vec{T}_u\|}$)

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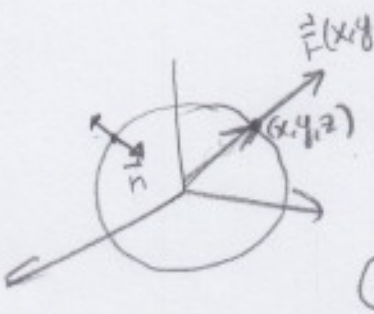
Ex: Let S be the sphere of radius 1 given by

$$\Phi(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$$

$$0 \le \theta \le 2\pi \qquad 0 \le \varphi \le \pi$$

$$= (x, y, z)$$



Consider the vector field

$$\vec{F}(x,y,z) = (x, y, z).$$

Compute the flux $\iint_S \vec{F}\cdot\vec{n}\, dS$, where $\vec{n}$ is the <u>inward</u> unit normal.

$$\vec{T}_\theta = (-\sin\theta\sin\varphi, \cos\theta, \sin\varphi, 0)$$

$$\vec{T}_\varphi = (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi).$$

$$\vec{T}_\theta \times \vec{T}_\varphi = (-\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi)$$

$$= -\sin\varphi\,(\underbrace{\sin\varphi\cos\theta}_{x}, \underbrace{\sin\varphi\sin\theta}_{y}, \underbrace{\cos\varphi}_{z}).$$

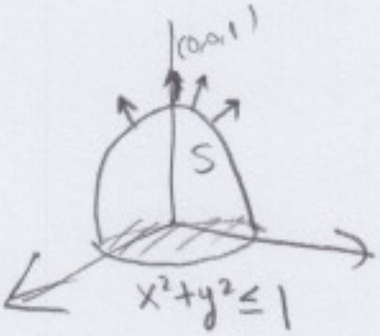
$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_D \vec{F}(\Phi(\theta,\varphi)) \cdot \frac{\vec{T}_\theta \times \vec{T}_\varphi}{\cancel{\|\vec{T}_\theta \times \vec{T}_\varphi\|}} \cdot \cancel{\|\vec{T}_\theta \times \vec{T}_\varphi\|}\, d\theta\, d\varphi$$

$$= \int_0^\pi \int_0^{2\pi} (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)\cdot(-\sin\varphi)(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)\, d\theta\, d\varphi$$

$$= \int_0^\pi \int_0^{2\pi} (-\sin\varphi)\, d\theta\, d\varphi = \int_0^\pi 2\pi(-\sin\varphi)\, d\varphi$$

$$= 2\pi\,[\cos\varphi]_0^\pi = 2\pi(-1-1) = \boxed{-4\pi}.$$

<u>Ex</u>: Let S be the paraboloid $z = 1-x^2-y^2$ above xy-plane oriented with normal upward. Let $\vec{F} = (x, y, 2z)$ be the velocity field of a fluid (m/sec). Compute how many cubic meters of fluid per second are crossing the surface.



$$\Phi(x,y) = (x, y, \overbrace{1-x^2-y^2}^{f(x,y)})$$

$$(x,y) \in D = \{x^2+y^2 \le 1\}.$$

$$T_x = \left(1, 0, \frac{\partial f}{\partial x}\right)$$

$$T_y = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

$$T_x \times T_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \vec{i}\left(-\frac{\partial f}{\partial x}\right) - \vec{j}\left(\frac{\partial f}{\partial y}\right) + \vec{k}$$

$$= \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

$$= (2x, 2y, 1) \checkmark$$

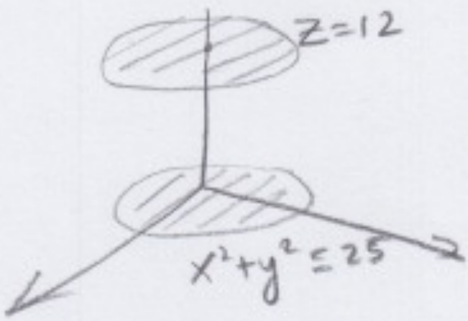
$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_{x^2+y^2\le 1} (x, y, 2-2x^2-2y^2) \cdot \frac{\vec{T}_x \times \vec{T}_y}{\cancel{\|\vec{T}_x \times \vec{T}_y\|}} \cancel{\|\vec{T}_x \times \vec{T}_y\|}\, dx\,dy$$

$$= \iint_{x^2+y^2\le 1} (x, y, 2-2x^2-2y^2)\cdot(2x, 2y, 1)\, dx\,dy$$

$$= \iint_{x^2+y^2\le 1} (\cancel{2x^2} + \cancel{2y^2} + 2 \cancel{-2x^2} \cancel{-2y^2})\, dx\,dy = 2\iint_{x^2+y^2\le 1} dx\,dy = 2\pi$$

Ex: Let S be the surface $z = 12$, over the domain $x^2+y^2 \le 25$, $\vec{F}(x,y,z) = (x,y,z)$.

Compute $\iint_S \vec{F}\cdot\vec{n}\, dS$, $\vec{n}$ upward normal



$$\Phi(x,y) = (x,y,12)$$
$$(x,y) \in D = \{x^2+y^2 \le 25\}$$

$$\vec{T}_x \times \vec{T}_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

$f(x,y) = 12$
$$= (0,0,1).$$

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_{x^2+y^2\le 25} (x,y,12)\cdot \frac{\vec{T}_x\times\vec{T}_y}{\cancel{\|\vec{T}_x\times\vec{T}_y\|}} \cancel{\|\vec{T}_x\times\vec{T}_y\|}\, dx\,dy$$

$$= \iint_{x^2+y^2\le 25} (x,y,12)\cdot(0,0,1)\, dx\,dy$$

$$= \iint_{x^2+y^2\le 25} 12\, dx\,dy = 12 \iint_{x^2+y^2\le 25} dx\,dy$$

$$\begin{array}{r} 25 \\ \underline{12} \\ 50 \\ \underline{25\ \ } \\ 300 \end{array}$$

$$= 12 \int_0^{2\pi}\int_0^5 r\, dr\,d\theta = 12(2\pi)\left[\frac{r^2}{2}\right]_0^5$$

$$= 12\pi(25) = 300\pi.$$