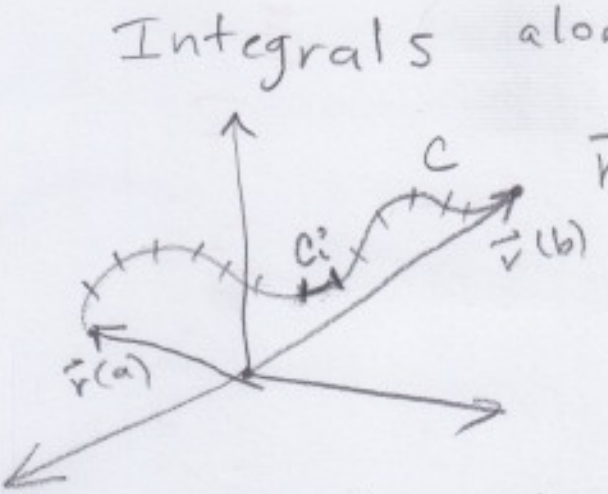
Section 7.1
Integrals along paths.

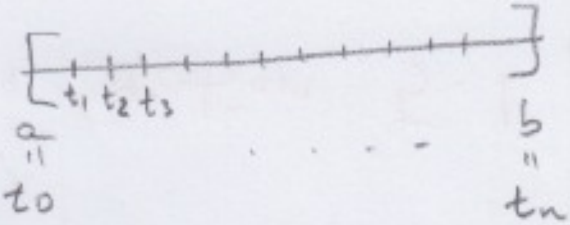


$\vec{r}(t) = (x(t), y(t), z(t))$

$a \le t \le b$

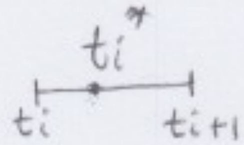
Goal: Integrate a scalar function $f: \mathbb{R}^3 \to \mathbb{R}$ along C. To fix ideas. assume $f(x,y,z)$. is the density in grs/unit length of wire. f is continuous

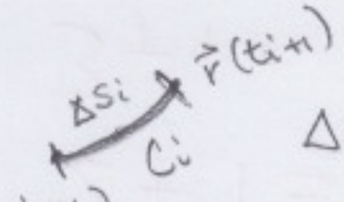
<u>Question:</u> Compute the mass of wire.



$\Delta t = t_{i+1} - t_i$

Mass = density × length





$$\Delta s_i = \int_{t_i}^{t_{i+1}} \|\vec{r}\,'(t)\|\, dt$$

$$= \|\vec{r}\,'(t_i^*)(t_{i+1} - t_i)$$

Mean Value theorem $= \|\vec{r}\,'(t_i^*)\|\, \Delta t$

Mass of $C_i \cong f(\vec{r}(t_i^*))\, \Delta s_i$

$$= f\,(x(t_i^*), y(t_i^*), z(t_i^*))\, \|\vec{r}\,'(t_i^*)\| \Delta t$$

$$\text{Mass of } C \cong \sum_{i=1}^{n} \text{Mass}(C_i)$$

$$\cong \sum_{i=1}^{n} f(x(t_i^*), y(t_i^*), z(t_i^*)) \|\vec{r}\,'(t_i^*)\| \Delta t$$

$$\text{Mass of } C = \lim_{\substack{n \to \infty \\ \Delta t \to 0}} \sum_{i=1}^{n} \underbrace{f(\vec{r}(t_i^*))}_{gr/\text{unit length}} \cdot \underbrace{\|\vec{r}\,'(t_i^*)\| \Delta t}_{\text{length}}$$

If is continuous, then the limit exists, and is denoted as $\int_C f\, ds$.

From Calculus 1, the limit is computed as:

$$\int_a^b f(\vec{r}(t)) \|\vec{r}\,'(t)\|\, dt$$

In practice we compute the limit $\int_C f\,ds$ as a Calculus 1 integral.

$\int_C f\,ds$ is called a path integral of f along C

$f: \mathbb{R}^3 \to \mathbb{R}$

Ex. Compute $\int_C f\,ds$ where (3)

$f(x,y,z) = \frac{x+y}{y-z}$ and C

is $\vec{r}(t) = (t, t^{3/2}, -t)$, $0 \le t \le 1$

$$\int_C f\,ds = \int_0^1 f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

$$\vec{r}\,'(t) = \left(1, \tfrac{3}{2}\sqrt{t}, -1\right)$$

$$\|\vec{r}\,'(t)\| = \sqrt{1+\tfrac{9}{4}t+1} = \sqrt{2+\tfrac{9}{4}t}$$

$$= \int_0^1 \frac{\cancel{t+t^{3/2}}}{\cancel{t^{3/2}-(-t)}} \sqrt{2+\tfrac{9}{4}t}\,dt$$

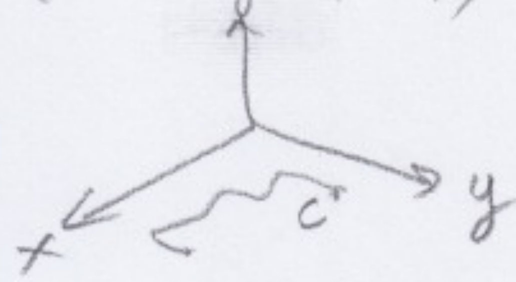
$$= \int_0^1 \sqrt{2+\tfrac{9}{4}t}\,dt = \frac{4}{9}\left.\frac{\left(2+\frac{9t}{4}\right)^{3/2}}{3/2}\right]_0^1$$

$$= \frac{4}{9}\cdot\frac{2}{3}\left.\left(2+\frac{9t}{4}\right)^{3/2}\right]_0^1$$

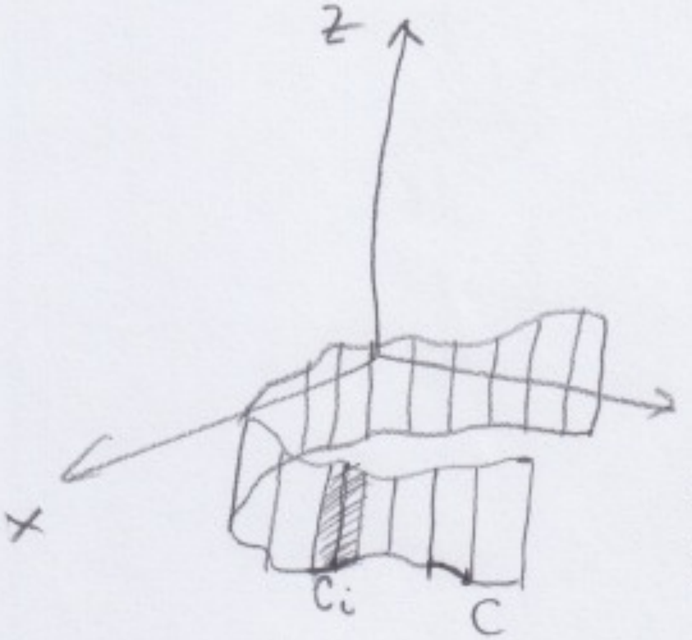
$$= \frac{8}{27}\left[\left(2+\frac{9}{4}\right)^{3/2} - 2^{3/2}\right] = \frac{8}{27}\left(\left(\frac{17}{4}\right)^{3/2} - 2^{3/2}\right)$$

Path integral for planar curves

$\vec{r}(t) = (x(t), y(t), 0)$, $f: \mathbb{R}^2 \to \mathbb{R}$



$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

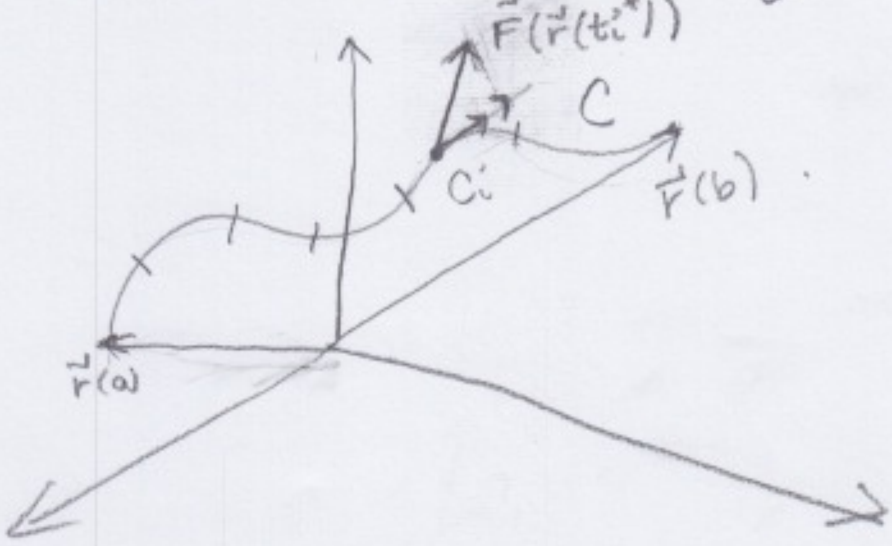


$C: \vec{r}(t) = (x(t), y(t), 0)$

$f(x,y)$

$\int_C f(x,y)\,ds$ = Area of fence.

Section 7.2 Line integrals



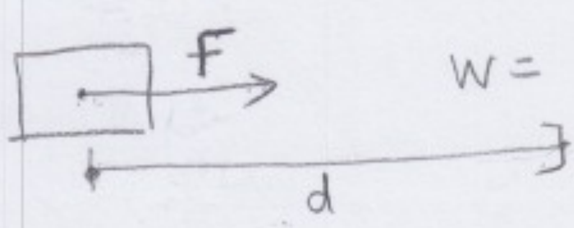
$$\frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Now, we want to integrate a vector field $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ over C

Q: Think of C as the path of a particle going from tip of $\vec{r}(a)$ to tip of $\vec{r}(b)$.

Assume $\vec{F}$ is the force acting on the particle. Compute work done by the particle.

Work = force × distance.



$W = F \times d$

$$W_i \cong \vec{F}(\vec{r}(t_i^*)) \cdot \frac{\vec{r}'(t_i^*)}{\|\vec{r}'(t_i^*)\|} \Delta s_i, \quad \Delta s_i = \|\vec{r}'(t_i^*)\| \Delta t$$

$$= \vec{F}(\vec{r}(t_i^*)) \cdot \frac{\vec{r}'(t_i^*)}{\|\vec{r}'(t_i^*)\|} \|\vec{r}'(t_i^*)\| \Delta t$$

⑥

$$W = \sum_{i=1}^{n} W_i$$

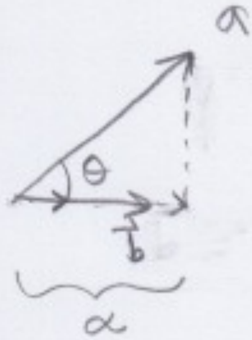
$$= \lim_{\substack{n \to \infty \\ \Delta t \to 0}} \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i^*)) \cdot \frac{\vec{r}'(t_i^*)}{\|\vec{r}'(t_i^*)\|} \|\vec{r}'(t_i^*)\| \Delta t$$

Calculus 1 $$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{\cancel{\|\vec{r}'(t)\|}} \cancel{\|\vec{r}'(t)\|} \, dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

Remark on orthogonal projection



$$\text{Proj}_{\vec{b}}\,\vec{a} = \alpha\,\frac{\vec{b}}{\|\vec{b}\|}$$

$$\alpha = \|\vec{a}\|\cos\theta$$

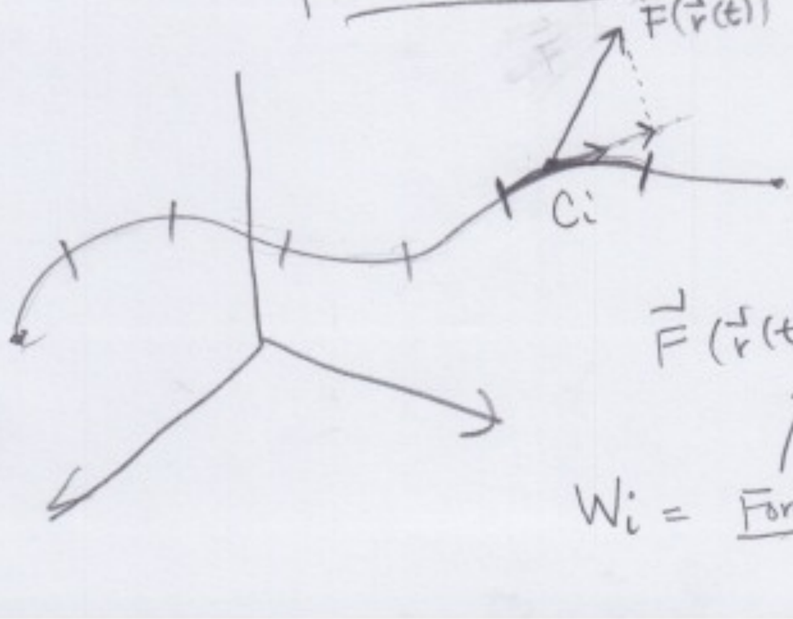
$$= \cancel{\|\vec{a}\|}\,\frac{\vec{a}\cdot\vec{b}}{\cancel{\|\vec{a}\|}\,\|\vec{b}\|} \quad ; \quad \vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

$$= \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$$

$$\text{Proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\cdot\frac{\vec{b}}{\|\vec{b}\|}$$

Observe that $\|\vec{b}\| = 1$ then

$$\boxed{\alpha = \vec{a}\cdot\vec{b}}$$



$\vec{F}(x,y,z)$.

$$\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$\vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$W_i = \underline{\text{Force}} \times \underbrace{\text{distance}}_{\Delta S_i}.$$