Section 1.2

Inner product or dot product

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. The inner product $\vec{a}, \vec{b}$ is defined as:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Lenght of a vector $\vec{a} = (a_1, a_2, a_3)$ is:

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

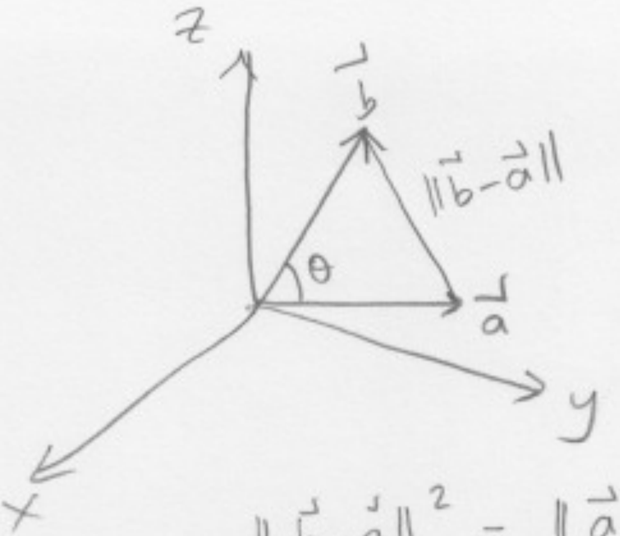
Notice that $\|a\|^2 = \vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$

The inner product of two vectors gives a measure of the angle between the vectors. We have the following:

<u>Theorem</u>: Let $\vec{a}$ and $\vec{b}$ be two vectors in $\mathbb{R}^3$ and let θ, $0 \le \theta \le \pi$ be the angle between them. Then:

$$\boxed{\vec{a} \cdot \vec{b} = \|a\| \, \|b\| \cos\theta.} \quad (*)$$

In order to see that $(*)$ is true, we use the law of cosines as follows:



$$\|\vec{b}-\vec{a}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

$$\therefore\ (b-a)\cdot(b-a) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

$$\cancel{\vec{b}\cdot\vec{b}} - 2\vec{a}\cdot\vec{b} + \cancel{\vec{a}\cdot\vec{a}} = \cancel{\vec{a}\cdot\vec{a}} + \cancel{\vec{b}\cdot\vec{b}} - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

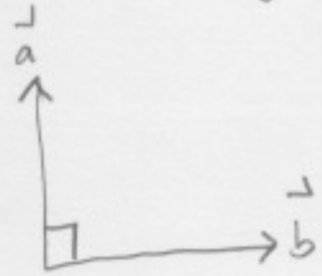
$$\Rightarrow\ \vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

Hence, we have the formula for dot product:

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

From this formula, note that:

$\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b}=0$



Ex: Compute θ if $\vec{u} = 5\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{v} = \vec{i} + \vec{j} - \vec{k}$.

We have:

$$\|\vec{u}\| = \sqrt{5^2 + (-1)^2 + (2)^2} = \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{(1)^2 + (1)^2 + (-1)^2} = \sqrt{3}$$

$$\Rightarrow \quad \vec{u} \cdot \vec{v} = \sqrt{30} \cdot \sqrt{3} \cdot \cos\theta$$

$$\vec{u} \cdot \vec{v} = (5)(1) + (-1)(1) + (2)(-1)$$
$$= 5 - 1 - 2 = 2$$

$$\therefore \quad \cos\theta = \frac{2}{\sqrt{3}\sqrt{30}}$$

$$\theta = \cos^{-1} \frac{2}{\sqrt{90}}.$$

The following inequality is called Cauchy-Schwarz inequality:

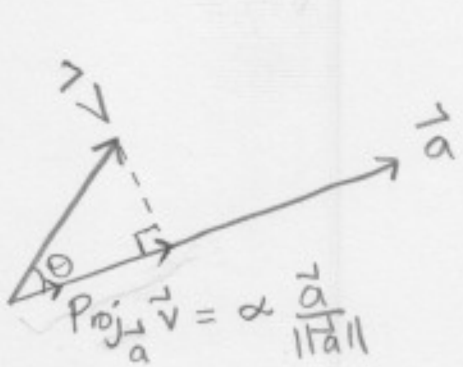
<u>Corollary</u>: Let $\vec{a}, \vec{b}$ two vectors, then:

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \, \|\vec{b}\|$$

The corollary is true, since $|\cos\theta| \leq 1$ and

$$|\vec{a} \cdot \vec{b}| = \|\vec{a}\| \, \|\vec{b}\| \, |\cos\theta| \leq \|\vec{a}\| \, \|\vec{b}\|$$

Orthogonal Projection



Let α be the magnitud of $\mathrm{Proj}_{\vec{a}}\vec{v}$, where $\mathrm{Proj}_{\vec{a}}\vec{v}$ is the orthogonal projection of $\vec{v}$ onto $\vec{a}$ (see picture).

We want to compute the vector $\mathrm{Proj}_{\vec{a}}\vec{v}$. The direction of $\mathrm{Proj}_{\vec{a}}\vec{v}$ is the same as the direction of $\vec{a}$. The vector $\frac{\vec{a}}{\|\vec{a}\|}$ has length 1, and has the same direction of $\vec{a}$. Therefore:

$$\mathrm{Proj}_{\vec{a}}\vec{v} = \alpha \frac{\vec{a}}{\|\vec{a}\|} = \|\vec{v}\| \cos\theta \frac{\vec{a}}{\|\vec{a}\|}$$

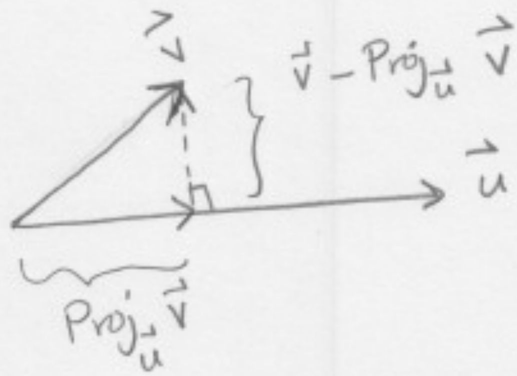
$$= \cancel{\|\vec{v}\|} \frac{\vec{a}\cdot\vec{v}}{\|\vec{a}\|\cancel{\|\vec{v}\|}} \frac{\vec{a}}{\|\vec{a}\|}$$

$$= \left(\frac{\vec{a}\cdot\vec{v}}{\|\vec{a}\|^2}\right)\vec{a}$$

Hence $\boxed{\mathrm{Proj}_{\vec{a}}\vec{v} = \left(\frac{\vec{a}\cdot\vec{v}}{\|\vec{a}\|^2}\right)\vec{a}}$

Given any vector $\vec{v}$, we can decompose $\vec{v}$ as the sum of two vectors:

$$\vec{v} = \text{Proj}_{\vec{u}}\vec{v} + (\vec{v} - \text{Proj}_{\vec{u}}\vec{v})$$



$\text{Proj}_{\vec{u}}\vec{v}$ is a vector parallel to $\vec{u}$

$\vec{v} - \text{Proj}_{\vec{u}}\vec{v}$ is orthogonal to $\vec{u}$.

Tringle inequality:

Theorem: If $\vec{a}$ and $\vec{b}$ are two vectors, then $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$.

Indeed:

$$\begin{aligned}\|\vec{a}+\vec{b}\|^2 &= (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) \\ &= \|\vec{a}\|^2 + 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2 \\ &= (\|\vec{a}\| + \|\vec{b}\|)^2\end{aligned}$$

Taking square roots on both sides:

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$