Section 2.3, continuation.

Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function at x_0, then the following limit holds:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \qquad (*)$$

If we let $x = x_0 + h$, we have from (*):

$$f'(x_0) = \lim_{h \to 0} \frac{f(x) - f(x_0)}{x - x_0}.$$

or:

$$\lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right] = 0$$

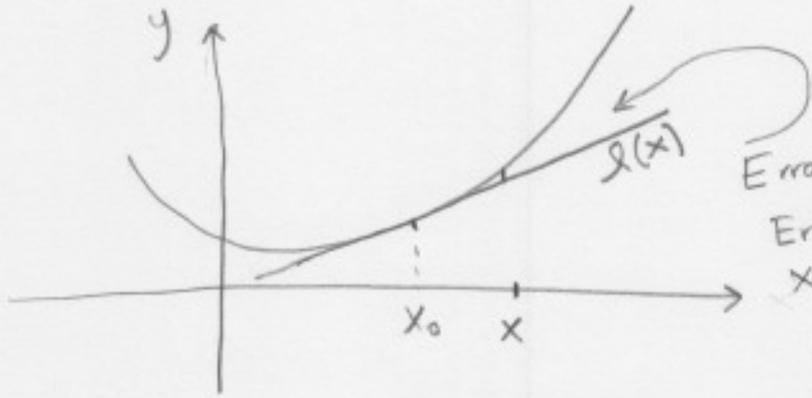
That is:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0$$

Thus:

$$\lim_{x \to x_0.} \frac{f(x) - \left[f(x_0) + f'(x_0)(x - x_0) \right]}{x - x_0} = 0$$

Note that $\ell(x) = f(x_0) + f'(x_0)(x - x_0)$ is the equation of the tangent line to the graph $y = f(x)$ at $(x_0, f(x_0))$. This limit being zero implies that $y = f(x)$ can be approximated with the tangent line $\ell(x)$ in a neighborhood of x_0.



Error $= f(x) - \ell(x)$
Error is small if $x - x_0$ is small.

<u>Def</u>: Let $f: U \subset \mathbb{R}^2 \to \mathbb{R}$, and $(x_0, y_0) \in U$. We say that f is differentiable at (x_0, y_0) if $\frac{\partial f}{\partial x}(x_0, y_0)$, $\frac{\partial f}{\partial y}(x_0, y_0)$ both exist <u>AND</u>:

$$\lim_{(x,y) \to (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}{\| (x,y) - (x_0,y_0) \|} = 0$$

If this limit is true, then $f(x,y)$ can be approximated with the equation of the tangent plane:

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0).$$

Ex: Approximate

$$(0.99\, e^{0.02})^8.$$

We define the function:

$$f(x,y) = (x e^{y})^8$$

and let $x_0 = 1$, $y_0 = 0$, $f(x_0, y_0) = f(1,0) = 1$.

Notice that 0.99 is close to 1 and 0.02 is close to 0.

We compute the eq. of the tangent plane:

$$\frac{\partial f}{\partial x} = 8\,(x e^{y})^7 e^{y} \qquad \frac{\partial f}{\partial x}(1,0) = 8$$

$$\frac{\partial f}{\partial y} = 8\,(x e^{y})^7 x e^{y} \qquad \frac{\partial f}{\partial y}(1,0) = 8$$

$$z = 1 + 8\,(x-1) + 8\,(y-0)$$

$$z = 1 + 8(x-1) + 8y$$

$$\Rightarrow f(0.99, 0.02) \cong 1 + 8\,(0.99 - 1) + 8(0.02)$$

$$= 1 + 8\,(-0.01) + 0.16$$

$$= 1 - 0.08 + 0.16$$

$$= 1.08$$

Differentiation for functions of n variables:

<u>Def</u>: Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$, where U is an open set in $\mathbb{R}^n$. With the notation:

$\vec{x} = (x_1, \dots, x_n)$, $\vec{x}_0 = (x_1^\circ, x_2^\circ, \dots, x_n^\circ)$ we define:

$$\frac{\partial f}{\partial x_j}(\vec{x}_0) = \lim_{h \to 0} \frac{f(x_1^\circ, \dots, x_j^\circ + h, \dots, x_n^\circ) - f(x_1^\circ, \dots, x_j^\circ, \dots, x_n^\circ)}{h}$$

In the most general situation we can have a function:

$$f: U \subset \mathbb{R}^n \to \mathbb{R}^m, \quad n, m > 1$$

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots x_n), \dots, f_m(x_1, \dots, x_n)),$$

where each $f_j(x_1, \dots, x_n) : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \dots, m$.

<u>Ex</u>: $n = 2$, $m = 2$

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(x, y) = (x e^{2y}, y x^2) = (f_1(x,y), f_2(x,y))$

$f_1(x,y) = x e^{2y}$, $f_2(x,y) = y x^2$

In this more general context, the notion of tangent plane is replaced by the matrix of partial derivatives:

$$T = Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

This is a $m \times n$ matrix.

Ex: Find Df for $f(x,y) = (xe^{2y}, yx^2)$

$$Df = \begin{pmatrix} \frac{\partial}{\partial x}(xe^{2y}) & \frac{\partial}{\partial y}(xe^{2y}) \\ \frac{\partial}{\partial x}(yx^2) & \frac{\partial}{\partial y}(yx^2) \end{pmatrix} = \begin{pmatrix} e^{2y} & 2xe^{y} \\ 2xy & x^2 \end{pmatrix}$$

Def: Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$. The function f is differentiable at $\vec{x}_0 \in U$ if all partial derivatives $\frac{\partial f_i}{\partial x_j}(\vec{x}_0)$ exist AND:

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\| f(\vec{x})^T - f(\vec{x}_0)^T - T(\vec{x} - \vec{x}_0)^T \|}{\| \vec{x} - \vec{x}_0 \|} = 0$$

where $T = Df(\vec{x}_0)$ and T means to "transpose" the vector (i.e, a column instead of a row).

Hence, the matrix T is an approximation of $f(\vec{x})$, if $\|\vec{x} - \vec{x}_0\|$ is small.

$$f(\vec{x})^T \cong \underset{m\times 1}{f(\vec{x}_0)^T} + \underbrace{\underset{m\times n}{T}\ \underset{n\times 1}{(\vec{x}-\vec{x}_0)^T}}_{m\times 1}$$

<u>Ex</u>: With $f(x,y) = (xe^{2y}, yx^2)$, $(x_0, y_0) = (1, 0)$.

$$Df(1,0) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Estimate $f(0.9, 0.2)$

$$f(0.9, 0.2) \cong \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0.9-1 \\ 0.2-0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -0.1 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.3 \\ 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 1.3 \\ 0.2 \end{pmatrix}$$

$$\therefore\ f(0.9, 0.2) \cong (1.3, 0.2)$$

The true value is $f(0.9, 0.2) = (0.9\,e^{0.4}, 0.2\,(0.9)^2)$

$$= (1.34\ldots, 0.162).$$

The most important case of Df is when $f : U \subset \mathbb{R}^n \to \mathbb{R}$. Then Df is a $1 \times n$ matrix. We can think of this matrix as a vector, called the gradient vectors, and denoted by ∇f.

<u>Ex</u>: $f(x,y,z) = (x e^{z^2} \sin y)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$= \left(e^{z^2} \sin y,\ x e^{z^2} \cos y,\ 2xz e^{z^2} \sin y \right)$$

<u>Remark</u>: Again, for $f : \mathbb{R}^n \to \mathbb{R}^m$, Df is a $m \times n$ matrix.

If $n = 1$ and $f : \mathbb{R} \to \mathbb{R}^m$, then Df is a $m \times 1$ matrix

<u>Ex</u>: $f : \mathbb{R} \to \mathbb{R}^3$, $f(t) = (t, t^2, t^3)$,

$$Df = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$$

If $m = 1$ and $f : \mathbb{R}^n \to \mathbb{R}$, then Df is a $1 \times n$ matrix, as in the example above:

$$Df = \left(e^{z^2} \sin y \quad x e^{z^2} \cos y \quad 2xz e^{z^2} \sin y \right).$$

We have the following important theorems:

<u>Theorem 1</u>: If f is differentiable at $\vec{x}_0$, then f is continuous at $\vec{x}_0$.

<u>Theorem 2</u>: Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose the partial derivatives $\frac{\partial f_i}{\partial x_j}$ of f all exist <u>and</u> are continuous in a neighborhood of a point $\vec{x} \in U$. Then f is differentiable at $\vec{x}$.

<u>Note</u>: If the hypothesis of Theorem 2 are true for every point $\vec{x}$ in the domain U, we say that f is of class C^1; that is, f is continuously differentiable in U.

Ex: Consider again the function:

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

We have seen that f is continuous in $\mathbb{R}^2$; in particular, f is continuous at 0.

But f is not differentiable at (0,0). We can see this by looking at the limit to check differentiability.

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \left[\cancelto{0}{f(0,0)} + \cancelto{0}{\frac{\partial f}{\partial x}(0,0)}x + \cancelto{0}{\frac{\partial f}{\partial y}(0,0)}y\right]}{\|(x,y)-(0,0)\|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2},$$

but this limit does not exist (check it!)

Also, the hipothesis of Theorem 2 do not hold at (0,0). Indeed, compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and check that these functions are not continuous at (0,0).