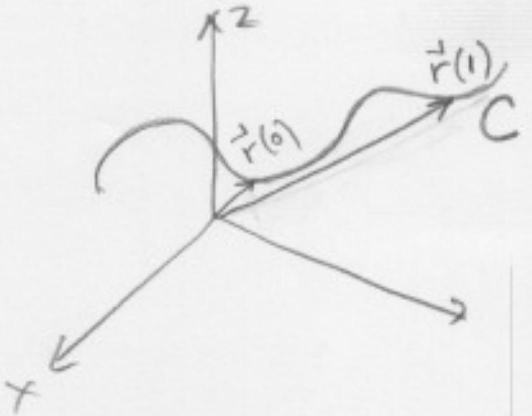
Section 2.4
Paths and curves.



We describe a curve C with a formula

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$\vec{r}(t)$ is called a path for the curve C

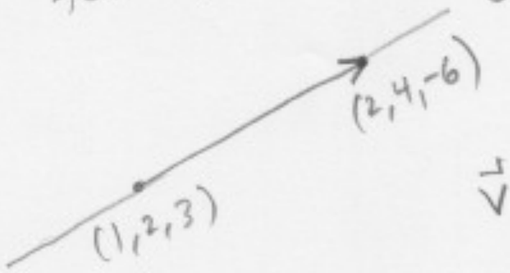
Notation: We use $\vec{r}(t)$, but sometimes we can use $\vec{R}(t)$ or $\vec{c}(t)$.

$\vec{r}(t)$ is a position vector for the curve. Its base point is at the origin, and the tip traces out the curve C.

Ex: We have already seen how to write a position vector for a line. Let (x_0, y_0, z_0) be a point on the line, and $\vec{v}$ a vector parallel to the line. Then, a path for the line is:

$$\vec{r}(t) = (x_0 + tv_1, y_0 + tv_2, z_0 + tv_3)$$

Ex: Find a parametric equation (path) for the line joining (1, 2, 3) and (2, 4, −6).



$$\vec{v} = (2-1, 4-2, -6-3) = (1, 2, -9)$$

$$\vec{r}(t) = (1+t,\ 2+2t,\ 3-9t),\ -\infty < t < \infty.$$

If we want an equation just for the segment between the two points, we would take $0 \le t \le 1$

Ex. The following is a helix:

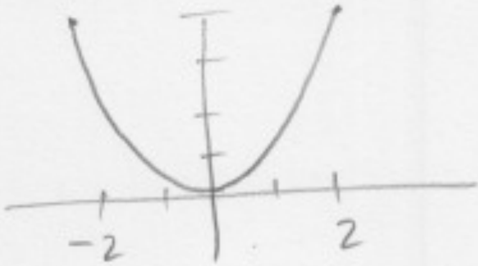
$$\vec{r}(t) = (\cos t,\ \sin t,\ t) \quad -\infty < t < \infty$$

Ex: If C is given by the graph of a function $y = f(x)$, $a \le x \le b$, we can always write a path for C as follows.

$$\vec{r}(t) = (t, f(t)),\ a \le t \le b$$

Ex: If C is the parabola $y = x^2$, $-2 \le x \le 2$, then a parametric equation is:

$$\vec{r}(t) = (t, t^2) \quad -2 \le t \le 2$$



$\vec{r}(-2) = (-2, 4)$
$\vec{r}(2) = (2, 4)$

* Given $\vec{r}(t) = (x(t), y(t), z(t))$, the velocity vector is:

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

In practice, we compute it as:

$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$$

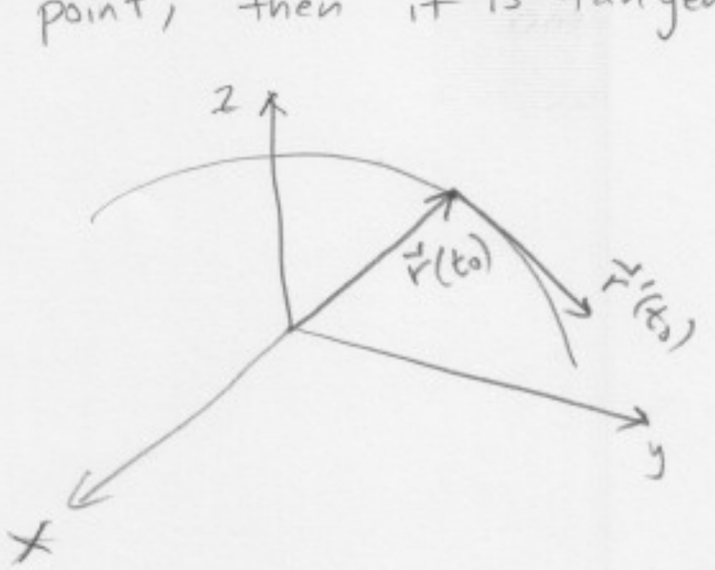
Remark: Notice that in the example above, we can change the speed of the curve. Another path is:

$$\vec{r}(t) = (t^3, t^6) \quad (-2)^{1/3} \le t \le 2^{1/3}$$

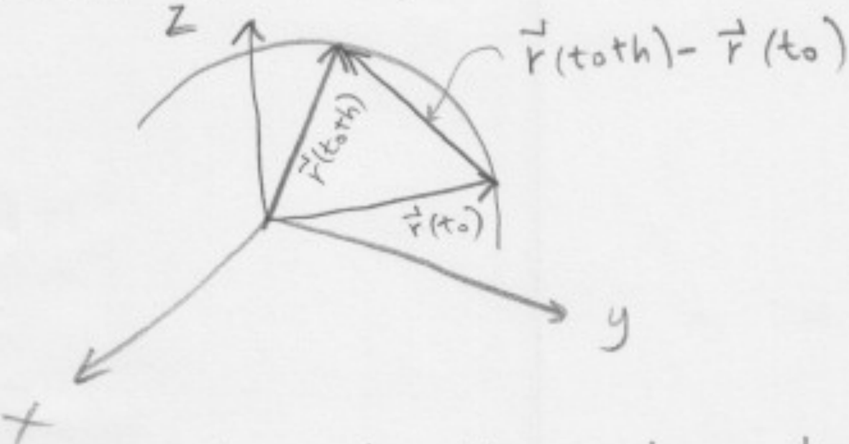
$$\vec{r}((-2)^{1/3}) = (-2, 4), \quad \vec{r}(2^{1/3}) = (2, 4)$$

Ex: For the helix $\vec{r}(t) = (\cos t, \sin t, t)$, the velocity vector is $\vec{r}\,'(t) = (-\sin t, \cos t, 1)$.

If the velocity vector $\vec{r}'(t)$ is not $\vec{0}$ at a point, then it is tangent to C



In order to see this, consider the picture.



$\vec{r}(t_0+h)-\vec{r}(t_0)$ has direction close to the tangent to C if $|h|$ is small, and $\dfrac{\vec{r}(t_0+h)-\vec{r}(t_0)}{h}$ is parallel to $\vec{r}(t_0+h)-\vec{r}(t_0)$.

<u>Def</u>: The speed of the path is $\|\vec{r}'(t)\|$.

We can use $\vec{r}'(t)$ as the direction vector $\vec{v}$ to find the tangent line to C.

Ex: Find the equation of the tangent line to the path $\vec{r}(t) = (\sec t, e^t, te^t)$, for $t=0$

$\vec{r}(0) = (1, 1, 0)$.
$\vec{r}\,'(t) = (\sec t \tan t, e^t, e^t + te^t)$
$\vec{r}\,'(0) = (0, 1, 1)$.

The eq. of the tangent line is:

$$\boxed{\vec{r}(t) = (1 + 0t, 1 + t, 0 + t) = (1, 1+t, t)}$$

Remark: Another solution is to consider

$$\vec{v} = (0, 2, 2)$$

and

$$\boxed{\vec{c}(t) = (1 + 0t, 1 + 2t, 0 + 2t) = (1, 1+2t, 2t)}$$

Both $\vec{r}(t)$ and $\vec{c}(t)$ yield the same line. The difference is that, if we think of a particle that travels along the line, the particle is traveling with different speed. We have $\vec{r}(1) = (1, 2, 1)$ and $\vec{c}(\frac{1}{2}) = (1, 2, 1)$, which means that the particle travels faster with $\vec{c}(t)$, in half the time the particle is at $(1, 2, 1)$. Indeed, since $\vec{r}\,'(t) = (0, 1, 1)$ and $\vec{c}\,'(t) = (0, 2, 2)$, the speeds are $\|\vec{r}\,'(t)\| = \sqrt{2}$ and $\|\vec{c}\,'(t)\| = 2\sqrt{2}$.