Section 3.1

Higher order derivatives

Notation: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^3 f}{\partial^2 y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right)$$

etc.

Other notations:

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$$

Ex: Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(x,y) = e^x \cos y$ is a solution. Let's check:

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$$

<u>Theorem</u>: Equality of mixed partials:

If $f(x,y)$ has continuous second partial derivatives (i.e., f is of class C^2), then:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Often in physics and engineering, one has to use Laplace's equation in polar coordinates.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In a previous class we computed $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in polar coordinates. We recall:

$u(r,\theta)$, $r(x,y) = (x^2+y^2)^{1/2}$, $\theta = \tan^{-1}\frac{y}{x}$, or
u of class C^2. $\tan\theta = \frac{y}{x}$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\Big(u(r(x,y), \theta(x,y))\Big)$$
$$= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\Big(u(r(x,y), \theta(x,y))\Big)$$
$$= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} \longrightarrow (2)$$

We compute:

$$r(x,y) = (x^2+y^2)^{1/2} \qquad \theta = \tan^{-1}\frac{y}{x} \quad \tan\theta = \frac{y}{x}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\sec^2\theta \, \frac{\partial\theta}{\partial x} = \frac{-y}{x^2} \Rightarrow \frac{\partial\theta}{\partial x} = \frac{-r\sin\theta}{r^2\cos^2\theta\,\sec^2\theta} = \frac{-\sin\theta}{r}$$

$$\sec^2\theta \, \frac{\partial\theta}{\partial y} = \frac{1}{x} \Rightarrow \frac{\partial\theta}{\partial y} = x\,\frac{1}{\sec^2\theta} = \frac{\cos\theta}{r}$$

$$\Rightarrow \boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} \end{aligned}}$$

$$\frac{\partial u}{\partial x}(r,\theta), \quad r(x,y) = (x^2+y^2)^{1/2} \quad \theta = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\partial u/\partial x\right)$$

$$= \frac{\partial}{\partial r}\left(\partial u/\partial x\right)\cdot\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\partial u/\partial x\right)\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cos\theta - \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\sin\theta}{r}$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\cos\theta - \frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta}{r} + \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r^2}\right)\cos\theta$$

$$- \left(\frac{\partial^2 u}{\partial \theta\partial r}\cos\theta - \frac{\partial u}{\partial r}\sin\theta - \frac{\partial^2 u}{\partial \theta^2}\frac{\sin\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\sin\theta}{r}$$

$$= \frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2} \cancel{- 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}} + \frac{1}{r}\frac{\partial u}{\partial r}\sin^2\theta$$

$$\cancel{+ 2\frac{\partial u}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2}}$$

$$\frac{\partial u}{\partial y}(r,\theta) \qquad r(x,y) = (x^2+y^2)^{1/2} \quad \theta = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\partial u/\partial y\right)$$

$$= \frac{\partial}{\partial r}\left(\partial u/\partial y\right)\cdot\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\left(\partial u/\partial y\right)\cdot\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\sin\theta$$

$$+ \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\cos\theta}{r}$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\sin\theta + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\cos\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r^2}\right)\sin\theta$$

$$+ \left(\frac{\partial^2 u}{\partial\theta\partial r}\sin\theta + \frac{\partial u}{\partial r}\cos\theta + \frac{\partial^2 u}{\partial \theta^2}\frac{\cos\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\cos\theta}{r}$$

$$= \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{\partial^2 u}{\partial \theta^2}\frac{\cos^2\theta}{r^2} + \cancel{2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}} + \frac{1}{r}\frac{\partial u}{\partial r}\cos^2\theta$$

$$\cancel{-2\frac{\partial u}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2}}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

Hence, the Laplace equation in polar coordinates is:

$$\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0}$$

<u>Ex</u>: Let $u(x,y)$, $x = x(s,t)$, $y(s,t)$.
Find a formula for $\frac{\partial^2 u}{\partial s^2}$.
u is of class C^2.
<u>Solution</u>:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s} \left(\partial u / \partial s \right)$$

$$= \frac{\partial}{\partial s} \left(\frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} \right)$$

$$= \frac{\partial x}{\partial s} \cdot \underline{\frac{\partial}{\partial s} \left(\partial u / \partial x \right)} + \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial s^2}$$

$$+ \frac{\partial y}{\partial s} \cdot \underline{\frac{\partial}{\partial s} \left(\partial u / \partial y \right)} + \frac{\partial u}{\partial y} \cdot \frac{\partial^2 y}{\partial s^2} \quad ; \quad \text{using product rule}$$

But

$$\frac{\partial}{\partial s} \left(\partial u / \partial x \right) = \frac{\partial}{\partial x} \left(\partial u / \partial x \right) \frac{\partial x}{\partial s} + \frac{\partial}{\partial y} \left(\partial u / \partial x \right) \cdot \frac{\partial y}{\partial s}$$

$$= \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y \partial x} \cdot \frac{\partial y}{\partial s} \qquad (1)$$

$$\frac{\partial}{\partial s} \left(\partial u / \partial y \right) = \frac{\partial}{\partial x} \left(\partial u / \partial y \right) \cdot \frac{\partial x}{\partial s} + \frac{\partial}{\partial y} \left(\partial u / \partial y \right) \cdot \frac{\partial y}{\partial s}$$

$$= \frac{\partial^2 u}{\partial x \partial y} \cdot \frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y^2} \cdot \frac{\partial y}{\partial s} \qquad (2)$$

Plugging (1) and (2) above we get:

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial x}{\partial s}\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y\,\partial x}\cdot\frac{\partial y}{\partial s}\right] + \frac{\partial u}{\partial x}\,\frac{\partial^2 x}{\partial s^2}$$

$$+ \frac{\partial y}{\partial s}\left[\frac{\partial^2 u}{\partial x\,\partial y}\cdot\frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y^2}\,\frac{\partial y}{\partial s}\right] + \frac{\partial u}{\partial y}\cdot\frac{\partial^2 y}{\partial s^2}$$

Hence

$$\boxed{\begin{aligned}\frac{\partial^2 u}{\partial s^2} = \left(\frac{\partial x}{\partial s}\right)^2\frac{\partial^2 u}{\partial x^2} + 2\,\frac{\partial^2 u}{\partial x\,\partial y}\,\frac{\partial x}{\partial s}\,\frac{\partial y}{\partial s} + \left(\frac{\partial y}{\partial s}\right)^2\frac{\partial^2 u}{\partial y^2} \\ + \frac{\partial u}{\partial x}\,\frac{\partial^2 x}{\partial s^2} + \frac{\partial u}{\partial y}\,\frac{\partial^2 y}{\partial s^2}\end{aligned}}$$

Ex: Let $u(x,y) = x^2 + y^2$, $x(s,t) = s^2 + t^2$, $y(s,t) = s + t$. Find $\frac{\partial^2 u}{\partial s^2}$

Using formula in square above:

$\frac{\partial x}{\partial s} = 2s \quad \frac{\partial y}{\partial s} = 1, \quad \frac{\partial^2 u}{\partial x^2} = 2 \quad \frac{\partial^2 u}{\partial y^2} = 2$

$\frac{\partial^2 x}{\partial s^2} = 2 \quad \frac{\partial^2 y}{\partial s^2} = 0 \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y,$ Plug:

$$\frac{\partial^2 u}{\partial s^2} = (2s)^2\cdot 2 + 2(0)(2s)(1) + (1)^2\cdot 2 + (2x)(2) + (2y)\cdot 0$$
$$= 8s^2 + 2 + 4x = 8s^2 + 2 + 4(s^2+t^2) = 8s^2 + 2 + 4s^2 + 4t^2$$

In particular:
$$\frac{\partial^2 u}{\partial s^2}(1,2) = 8(1) + 2 + 4(1) + 4(4) = 30 .$$

Since we have formulas for x and y in terms of s and t, we could obtain $\frac{\partial^2 u}{\partial s^2}$ in a more direct way as follows:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$
$$= (2x)(2s) + (2y)(1)$$
$$= 4s(s^2+t^2) + 2(s+t)$$

$$\Rightarrow \frac{\partial^2 u}{\partial s^2} = 4(s^2+t^2) + 4s(2s) + 2$$
$$= \underline{4s^2 + 4t^2 + 8s^2 + 2} \quad \checkmark \text{ (same result).}$$

We could also do:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$
$$= (2x)(2s) + (2y)(1), \quad x = x(s,t), \; y = y(s,t)$$

$$\frac{\partial^2 u}{\partial s^2} = 2s\left(2\frac{\partial x}{\partial s}\right) + (2x)(2) + 2\frac{\partial y}{\partial s}$$
$$= 4s \cdot (2s) + 4(s^2+t^2) + 2(1)$$
$$= \underline{8s^2 + 4s^2 + 4t^2 + 2} \quad \checkmark \text{ (same result)}$$

Ex: Let $f(x,y)$

$$x = x(u,v) = u+v$$
$$y = y(u,v) = u-v)$$

Let:

$$w(u,v) = f(u+v,\, u-v).$$

Compute $\dfrac{\partial^2 w}{\partial v \partial u}$.

Solution:

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}(u+v, u-v)\cdot 1 + \frac{\partial f}{\partial y}(u+v, u-v)\cdot 1$$

$$\frac{\partial^2 w}{\partial v \partial u} = \frac{\partial}{\partial v}\left[\frac{\partial f}{\partial x}(u+v, u-v) + \frac{\partial f}{\partial y}(u+v, u-v)\right]$$

$$= \frac{\partial^2 f}{\partial x^2}(u+v, u-v)\cdot 1 + \cancel{\frac{\partial^2 f}{\partial y \partial x}(u+v, u-v)\cdot(-1)}$$

$$+ \cancel{\frac{\partial^2 f}{\partial x \partial y}(u+v, u-v)\cdot 1} + \frac{\partial^2 f}{\partial y^2}(u+v, u-v)\cdot(-1)$$

$$= \frac{\partial^2 f}{\partial x^2}(u+v, u-v) - \frac{\partial^2 f}{\partial y^2}(u+v, u-v). \quad \blacksquare$$