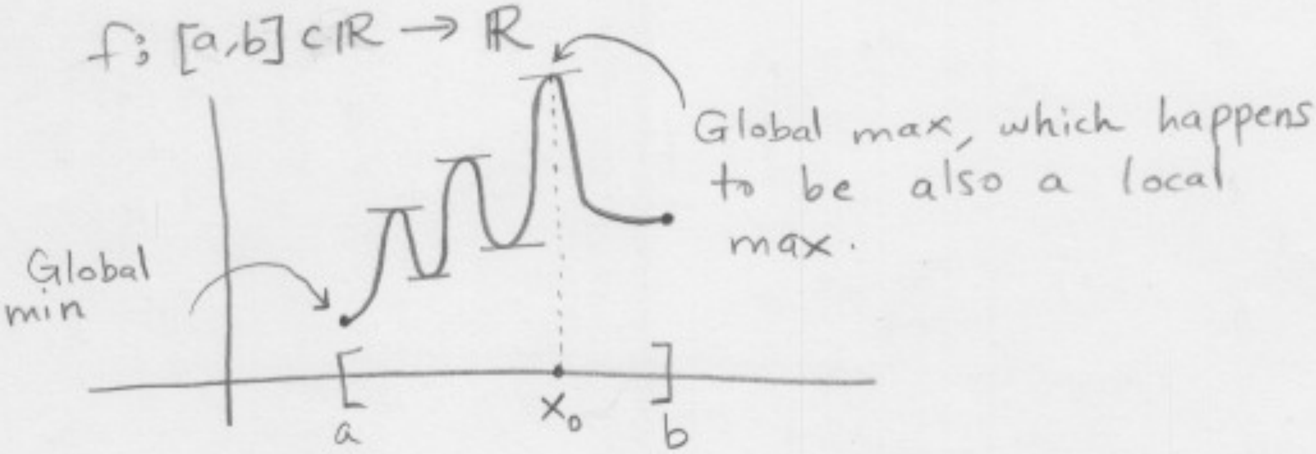
Section 3.4

Global extrema and Lagrange multipliers.

Last class we looked at local extrema. Now we consider global extrema. This means that for $f: D \subset \mathbb{R}^n \to \mathbb{R}$, we want to find the largest possible value and smallest possible value for f in the domain D.

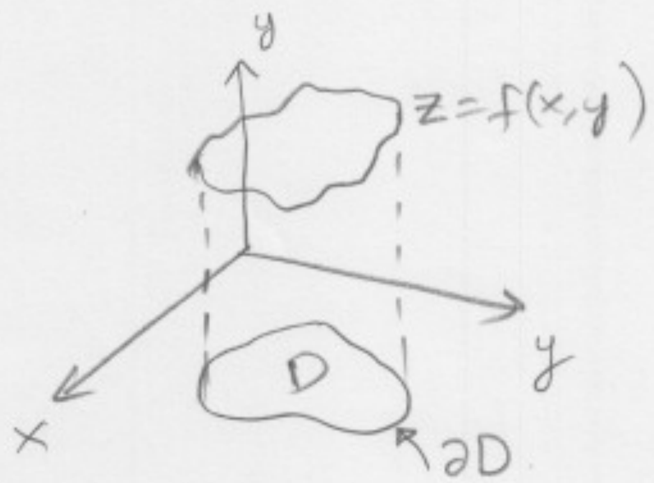
In one-dimension:



In the picture you see 3 local max and 2 local min. However, the global min is attained at the boundary of the domain $D = [a,b]$. Indeed, the global min is attained at $x=a$, and the global max is attained at $x=x_0$. Note that at $x=a$, f does not have a local min, while at $x=x_0$ we have both local max and global max.

We will frequently refer to the following theorem in order to justify the existence of global max and min.

<u>Theorem (***)</u>: If D is a closed bounded set in $\mathbb{R}^n$ and $f: D \to \mathbb{R}$ is continuous, then f has a global max and min in D.

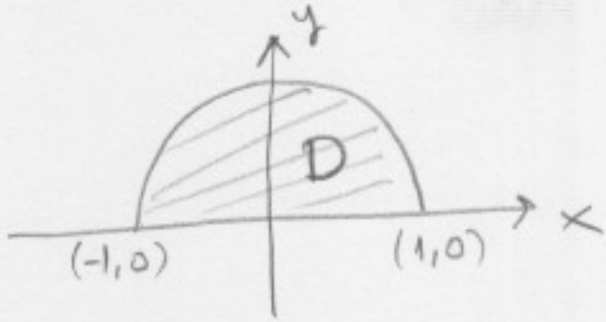


Procedure to find Global extrema.

1.- Find critical points in the interior of D and determine if they are local max, min or neither.

2.- Check the value of f on ∂D and compare with local extrema. (Note: When we check ∂D, we apply 1.- and 2.- to a function of one variable).

<u>Ex</u>: Find the global max and min of $f(x,y) = 16x^2 - 24xy + 40y^2$ on the closed half disk $x^2+y^2 \le 1$, $y \geq 0$.



∂D contains two pieces: the line joining $(-1,0)$ and $(1,0)$, and the upper arch of the unit disk.

1.- $\nabla f = (32x - 24y, -24x + 80y) = \vec{0}$

$$\left.\begin{aligned} 32x - 24y &= 0 \\ -24x + 80y &= 0 \end{aligned}\right\}$$ Two lines that pass through the origin.

The only intersection of these 2 lines is $(0,0)$. But $(0,0)$ is not in the interior of D. Thus, we say that for step 1, there are no interior critical points. The point $(0,0)$ will appear later when checking ∂D.

2.- We check now ∂D. We look first at the line. If $y=0$, we define:

$$g(x) := f(x,0) = 16x^2, \quad -1 \le x \le 1$$

We now apply steps 1- and 2- to g:

$$g'(x) = 32x$$
$$g'(0) = 0$$
$$g''(x) = 32 > 0 \Rightarrow x = 0 \text{ local min}$$

with value:

$$\boxed{g(0) = f(0,0) = 0} \; (A)$$

We now check the boundary $(-1,0)$ and $(0,1)$:

$$(B) \; \boxed{g(-1) = f(-1,0) = 16}, \quad \boxed{g(1) = f(1,0) = 16} \; (C)$$

For the circular part, we let $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le \pi$, we define:

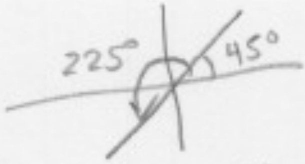
$$g(\theta) = 16\cos^2\theta - 24\cos\theta\sin\theta + 40\sin^2\theta$$

We apply steps 1- and 2- to g.

$$g'(\theta) = -32\cos\theta\sin\theta - 24\cos^2\theta + 24\sin^2\theta + 80\sin\theta\cos\theta$$
$$= 48\sin\theta\cos\theta - 24(\cos^2\theta - \sin^2\theta)$$
$$= 24\sin 2\theta - 24\cos 2\theta = 0$$
$$\cos 2\theta = \sin 2\theta$$
$$\tan 2\theta = 1$$

$\tan 2\theta = 1$

$\Rightarrow \quad 2\theta = \frac{\pi}{4}, \frac{5\pi}{4}$



$$\begin{array}{r} 180 \\ +\ 45 \\ \hline 225 \end{array}$$

$\Rightarrow \theta = \frac{\pi}{8}, \frac{5\pi}{8}$

$\cancel{225}°\left(\frac{\pi}{\cancel{180°}}\right) = \frac{5\pi}{4}$

$\frac{\pi}{8} = 22.5° \qquad \frac{5\pi}{8} = 112.5°$

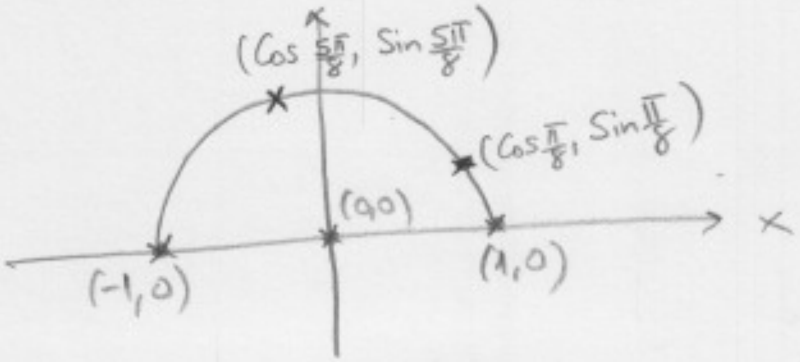
$$\boxed{g\left(\frac{\pi}{8}\right) = f\left(\cos\frac{\pi}{8}, \sin\frac{\pi}{8}\right) \simeq 11} \ (D)$$

$$\boxed{g\left(\frac{5\pi}{8}\right) = f\left(\cos\frac{5\pi}{8}, \sin\frac{5\pi}{8}\right) \approx 45} \ (E)$$

We also need to check the endpoints of the arch, but they were already checked since they are the same endpoints of the line.

$g(0) = f(\cos 0, \sin 0) = f(1, 0) = 16$
$g(\pi) = f(\cos\pi, \sin\pi) = f(-1, 0) = 16$ (already checked)

Therefore, we have 5 competitors for global max and min, squares (A), (B), (C), (D), (E).



We conclude

- Global max at $\left(\cos\frac{5\pi}{8}, \sin\frac{5\pi}{8}\right) \approx (-0.38, 0.9)$ with value ≈ 45.
- Global min at $(0,0)$ with value 0.

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Lagrange multipliers

Let $f(x,y,z): \mathbb{R}^3 \to \mathbb{R}$ and let S be the level surface $g(x,y,z) = c$.

Problem: Find the global max and min of $f(x,y,z)$ when restricted to the surface S.

Ex: $f(x,y,z) = x+z$
and S is the sphere $g(x,y,z) = 1$, where $g(x,y,z) = x^2+y^2+z^2$.

Thus, we are testing the values of f but only on points of the sphere. For example, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ belong to the sphere. And:

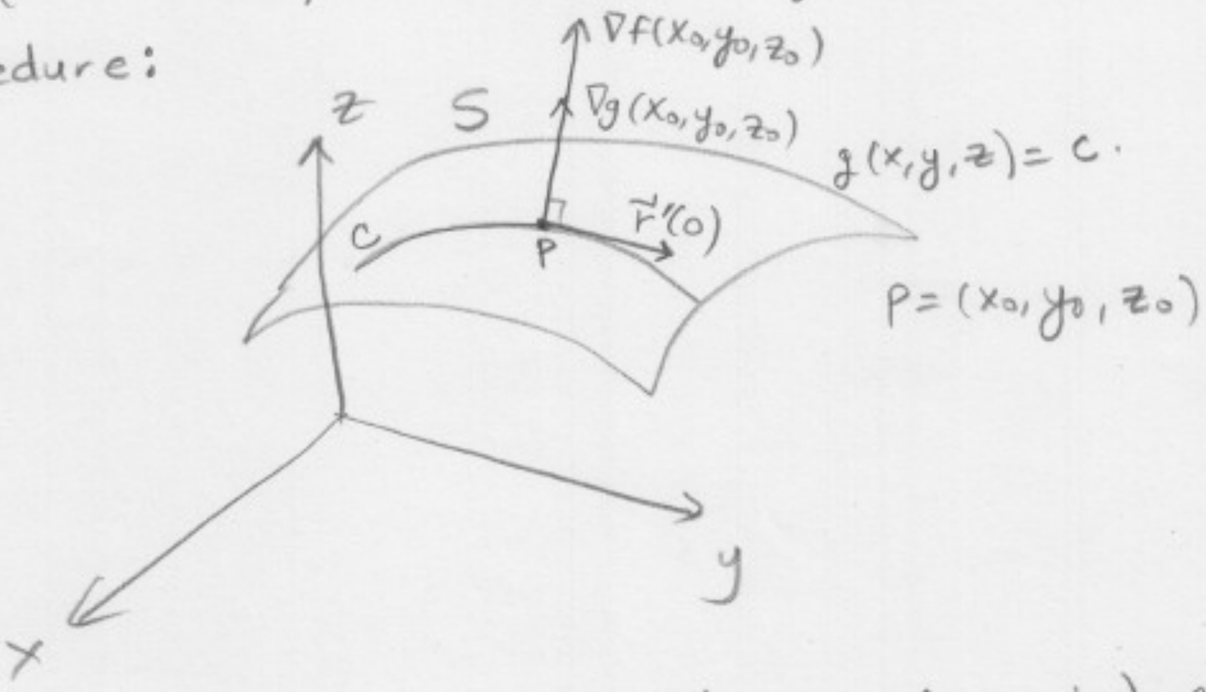
$$f(1,0,0) = 1+0 = 1$$
$$f(0,1,0) = 0+0 = 0$$
$$f(0,0,1) = 0+1 = 0$$
$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

But $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$ also belong to the sphere, and

$$f\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right) = \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} = \tfrac{2}{\sqrt{2}} > \tfrac{2}{\sqrt{3}} = f\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)$$

and $f\left(-\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right) = -\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} = \tfrac{-2}{\sqrt{2}}$

We need a procedure to determine if we can do better than $\frac{2}{\sqrt{2}}$ (for a max) or $-\frac{2}{\sqrt{2}}$ (for a min). We now analyze this procedure:



Suppose f has a global max (or min) at $P = (x_0, y_0, z_0)$. Let C be a curve in S which passes through (x_0, y_0, z_0) at $t = 0$. Let $\vec{r}(t) = (x(t), y(t), z(t))$ be the parametrization of C, $\vec{r}(0) = (x_0, y_0, z_0)$. We form the composition:

$$f(\vec{r}(t)) : \mathbb{R} \to \mathbb{R}.$$

Note that $f(\vec{r}(t))$ has a local max (or min) at $t = 0$. Therefore:

$$\frac{d}{dt} f(\vec{r}(t))\Big|_{t} = 0$$

$$\Rightarrow \quad \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\Big|_{t=0} = 0$$

$$\Rightarrow \quad \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0) = 0$$

$$\Rightarrow \quad \nabla f(x_0, y_0, z_0) \cdot \vec{r}\,'(0) = 0.$$

Hence $\nabla f(x_0, y_0, z_0)$ is perpendicular to the velocity vector $\vec{r}\,'(0)$. Since C is arbitrary (i.e, we can choose any other C and the same conclusion applies) it follows that $\nabla f(x_0, y_0, z_0)$ is perpendicular to S; that is, $\nabla f(x_0, y_0, z_0)$ is perpendicular to the tangent plane.

From section 2.6, we have that $\nabla g(x_0, y_0, z_0)$ is perpendicular to S. Hence

$$\nabla f(x_0, y_0, z_0) \perp S$$

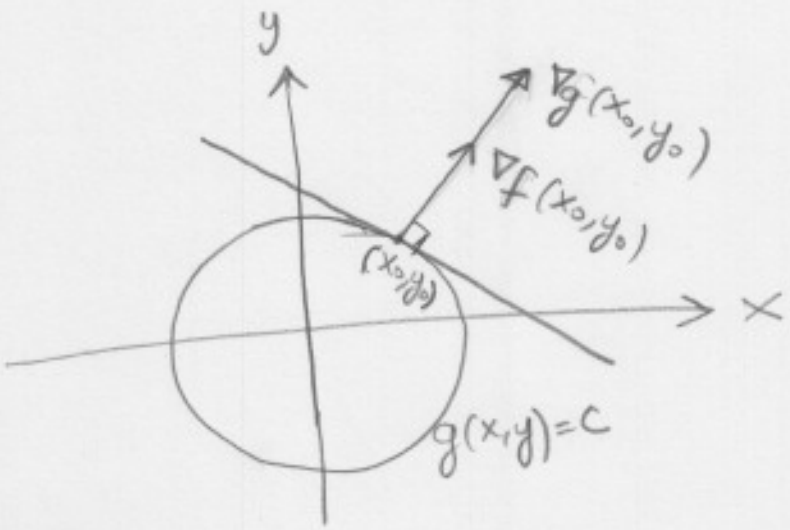
$$\nabla g(x_0, y_0, z_0) \perp S$$

Hence, $\nabla f(x_0, y_0, z_0)$ is parallel to $\nabla g(x_0, y_0, z_0)$ and thus, there exists $\lambda > 0$ such that

$$\boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)}$$

<u>Remark</u>: The same argument holds in 2-d. If $f(x,y): \mathbb{R}^2 \to \mathbb{R}$ has a global max (or min) at (x_0, y_0) when restricted to the level curve $g(x,y) = c$ then:

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0), \text{ for some } \lambda > 0$$



<u>Conclusion</u>: In order to solve:

Min/max $f(x,y,z)$ subject to $g(x,y,z) = c$

we need to solve:

Find all (x,y,z) that solve these 4 equations. The global max/min will be among these points.

$$\begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\ \dfrac{\partial f}{\partial z} = \lambda \dfrac{\partial g}{\partial z} \\ g(x,y,z) = c \end{cases}$$

* In 2-d:

Find global max/min of $f(x,y)$ subject to $g(x,y)=c$. Need to solve:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\ g(x,y) &= c \end{aligned}\right\}$$

Find all points (x,y) that solve these 3 equations. The max/min will be among these points.

* In $\mathbb{R}^n$:

Find the global max/min of $f(x_1,\dots,x_n)$ subject to $g(x_1,\dots,x_n)=c$. Need to solve:

$n+1$ equations.
$$\begin{cases} \frac{\partial f}{\partial x_1}(\vec{x}) = \lambda \frac{\partial g}{\partial x_1} \\ \frac{\partial f}{\partial x_2}(\vec{x}) = \lambda \frac{\partial g}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n}(\vec{x}) = \lambda \frac{\partial g}{\partial x_n} \\ g(\vec{x}) = c \end{cases}$$

We can now solve the example in page 130:

Find max/min of $f(x,y,z)=x+z$ on the sphere $x^2+y^2+z^2=1$.

Here, $g(x,y,z)=x^2+y^2+z^2$

$$\nabla f = \lambda \nabla g$$

$$(1, 0, 1) = \lambda (2x, 2y, 2z)$$

$$\begin{cases} ① & 1 = 2\lambda x \\ ② & 0 = 2\lambda y \\ ③ & 1 = 2\lambda z \\ ④ & x^2+y^2+z^2=1 \end{cases}$$

From ① $\Rightarrow \lambda \neq 0$

Since $\lambda \neq 0$, from ② $\Rightarrow y = 0$

From ① and ③ $\Rightarrow x = z$

From ④: $x^2 + x^2 = 1$, $2x^2 = 1$, $x = \pm \frac{1}{\sqrt{2}}$

We found: $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

We recall that Theorem (***) guarantees the existence of an absolute max and min on the sphere, since f is continuous and the sphere is a closed set in $\mathbb{R}^3$. Hence

$$f\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} \quad (\text{max})$$

$$f\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = \frac{-2}{\sqrt{2}} \quad (\text{min})$$