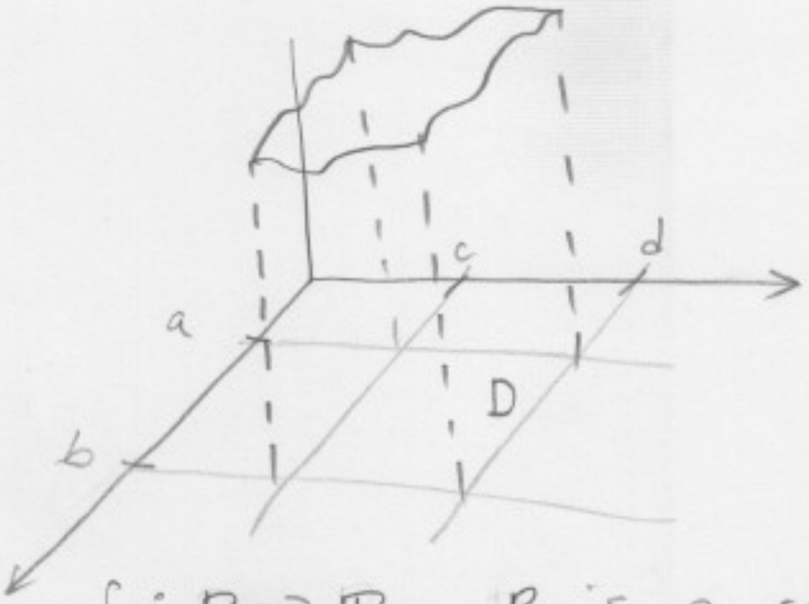
Section 5.1 Double Integral.

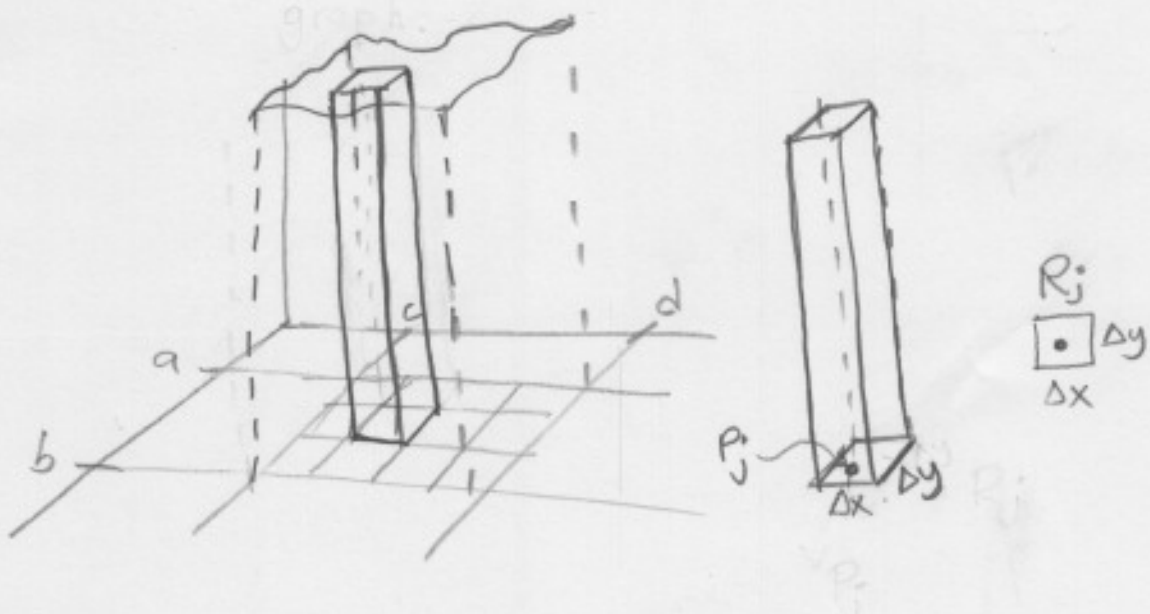


$f: R \to \mathbb{R}$, R is a rectangle

$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

or $R = [a,b] \times [c,d]$, f continuous on D

We want to compute the volume under the graph



We fix n and we partition D in to n rectangles $R_1, R_2, \ldots, R_n$.

The volume of the parallelepiped over R_j with height $f(P_j)$ is:

$$f(P_j)\Delta x \Delta y$$

and we use this to approximate the volume.

We define:

$$S_n := \sum_{j=1}^{n} f(P_j)\Delta x \Delta y$$

S_n is very close to the true volume for n large.

<u>Theorem</u>: If f is continuous on D then

$$\lim_{n\to\infty} S_n$$

exists and is independent of the choice of points P_j.

We denote the limit as S, and hence the volume under the graph is S.

This limit S is denoted by:

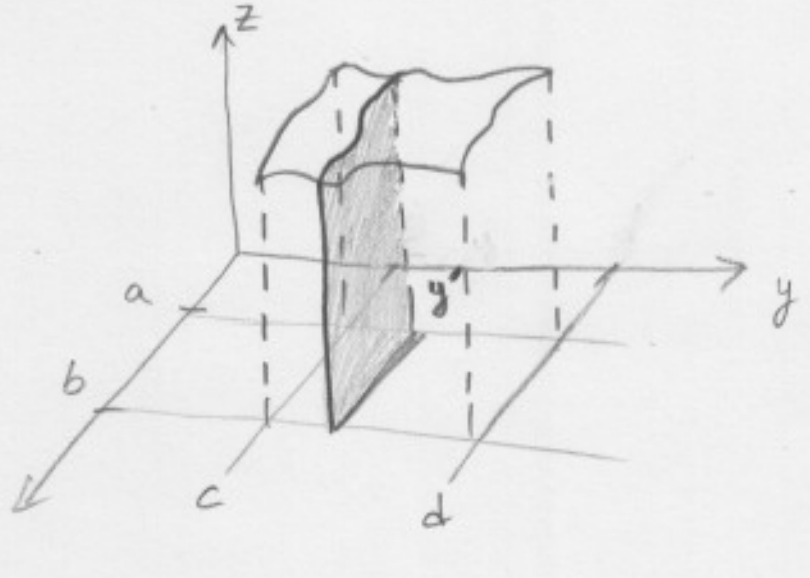
$$\iint_R f(x,y)\, dA = \iint_R f(x,y)\, dx\, dy.$$

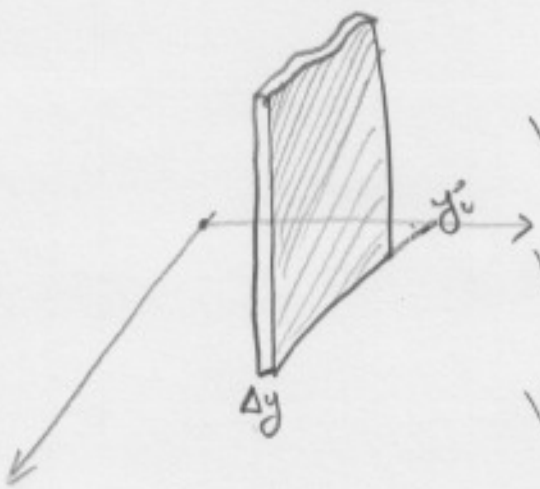
How to compute in practice $\iint_R f(x,y)\,dA$.

Fubini's theorem shows that:

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

Let $A(y) = \int_a^b f(x,y)\,dx$, we have that $A(y)$ is the area of the slice in this picture.





$$V_i = \left(\int_a^b f(x, y_i)\,dx \right) \Delta y$$

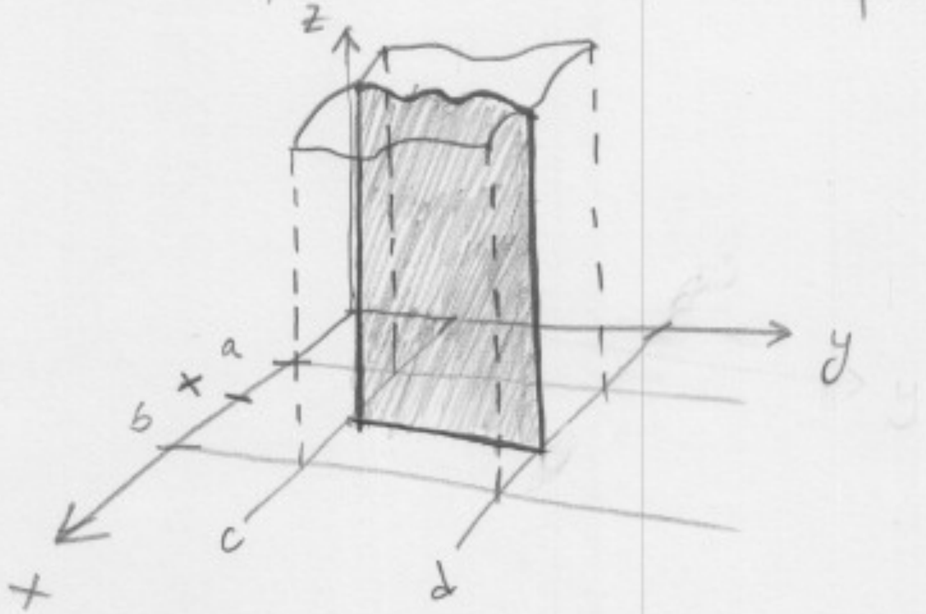
$$V \approx \sum_{i=1}^{n} V_i$$

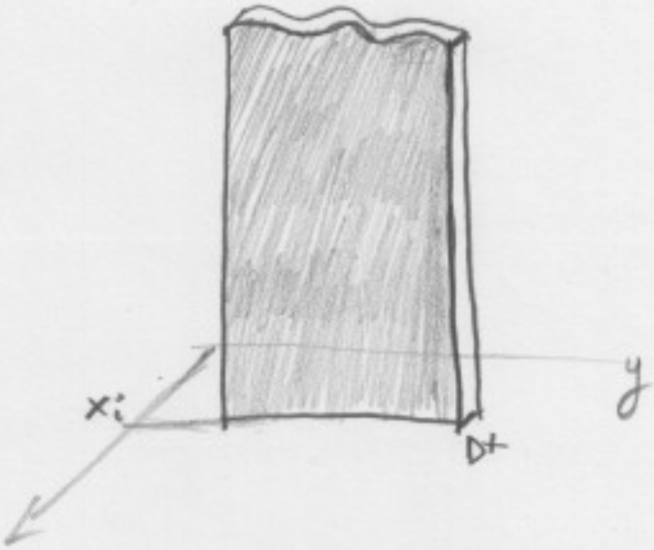
$$V = \lim_{\Delta y \to 0} \sum_{i=1}^{n} V_i = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

We can also compute V as follows:

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

Let $A(x) = \int_c^d f(x,y)\,dy$, we have that $A(x)$ is the area of the slice in this picture.





$$V_i = \left(\int_c^d f(x_i, y)\,dy \right) \Delta x$$

$$V \cong \sum_{i=1}^{n} V_i$$

$$V = \lim_{\Delta x \to 0} \sum_{i=1}^{n} V_i = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Ex: Compute the volume above $[0,1]\times[0,\pi]$ and below $z = y\sin xy + 1$

$$\int_0^\pi \left(\int_0^1 (y \sin xy + 1)\ dx \right) dy$$

$$= \int_0^\pi \left[-\cos xy + x \right]_0^1 dy$$

$$= \int_0^\pi \left[-\cos y + 1 - (-1) \right] dy$$

$$= \int_0^\pi (2 - \cos y)\, dy = \left[2y - \sin y \right]_0^\pi = 2\pi.$$

Ex: $R = [0,1]\times[0,1]$

Compute $\iint\limits_R x \ln(1+xy)\, dA$

We can try $\int_0^1\int_0^1 x\ln(1+xy)\,dx\,dy$ or $\int_0^1\int_0^1 x \ln(1+xy)\,dy\,dx$.

Recall that $\int \ln t\, dt = t\ln t - t + C$

$u = \ln t \quad dv = dt \qquad \int u\,dv = uv - \int v\,du$

$du = \frac{dt}{t} \quad v = t \qquad = t\ln t - \int dt$

$= t\ln t - t + C$

Then:

$$\int_0^1 \left[\int_0^1 x \ln(1+xy)\, dy \right] dx \qquad \begin{array}{l} t = 1+xy \\ dt = x\,dy \end{array}$$

$$= \int_0^1 \left[(1+xy) \ln(1+xy) - (1+xy) \right]_{y=0}^{y=1} dx$$

$$= \int_0^1 \left[(1+x) \ln(1+x) - (1+x) + 1 \right] dx$$

$$= \int_0^1 \left[(1+x) \ln(1+x) - x \right] dx = \int_0^1 (1+x) \ln(1+x)\, dx - \int_0^1 x\, dx$$

$t = 1+x \qquad dt = dx$

$u = \ln t \qquad dv = t\, dt$

$du = \frac{1}{t} \qquad v = \frac{1}{2} t^2$

$$\int u\, dv = uv - \int v\, du = \frac{1}{2} t^2 \ln t - \int \frac{1}{2} t\, dt$$

$$= \frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 + C$$

$$= \left[\frac{1}{2} (1+x)^2 \ln(1+x) - \frac{1}{4} (1+x)^2 \right]_0^1 - \left[\frac{1}{2} x^2 \right]_0^1$$

$$= 2 \ln 2 - 1 + \frac{1}{4} - \frac{1}{2}$$

$$= 2 \ln 2 - \frac{5}{4}$$