Another Version of Riesz Representation Theorem

RRT3: Local Version

Let $L: C_c(\mathbb{R}^n;\mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying

$$(*) \quad \sup\{L(f) \mid f \in C_c(\mathbb{R}^n;\mathbb{R}^m),\ |f| \leq 1,\ \operatorname{Spt}(f) \subset K\} < \infty$$

for each compact set $K \subset \mathbb{R}^n$. Then there exists a Radon outer measure μ on $\mathbb{R}^n$ and a μ-measurable function $\sigma: \mathbb{R}^n \to \mathbb{R}^m$ such that:

(i) $|\sigma(x)| = 1$ for μ-a.e. x.

(ii) $L(f) = \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu$, $\forall f \in C_c(\mathbb{R}^n;\mathbb{R}^m)$.

RRT3: Global Version.

Consider $C_0(\mathbb{R}^n)$, the completion of the space $C_c(\mathbb{R}^n)$ (which is endowed with the norm $\|f\| = \sup\{|f(x)| : x \in \mathbb{R}^n\}$). Let:

$\mathcal{M}(\mathbb{R}^n) = \{\mu : \mu$ is a Borel regular signed measure, $|\mu|(\mathbb{R}^n) < \infty\}$.

$C_0(\mathbb{R}^n)$ is a Banach space with the norm:

$$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}^n\}.$$

Then the map:

$$\Psi : \mathcal{M}(\mathbb{R}^n) \longrightarrow (C_0(\mathbb{R}^n))^*$$

8.78

given by

$$\Psi(\mu) = \int_{\mathbb{R}^n} f \, d\mu, \quad \forall f \in C_0(\mathbb{R}^n)$$

is an isometric isomorphism. That is:

$$\|\mu\| = |\mu|(\mathbb{R}^n) = \|\Psi(\mu)\|$$

$$= \sup \left\{ \int_{\mathbb{R}^n} f \, d\mu : f \in C_0(\mathbb{R}^n), \; \|f\| \leq 1 \right\}.$$

Similar version can be proven for the cases

$C_0(X)$, $C_c(X)$, X locally compact separable metric space

or

$C_0(X)$, $C_c(X)$, X locally compact Hausdorff space

Proof of RRT3, Local version

Step 1: Define an outer measure μ in $\mathbb{R}^n$.

Let $V \subset \mathbb{R}^n$ open. Define:

$$\mu(V) := \sup\{ L(f) : f \in C_c(\mathbb{R}^n; \mathbb{R}^m), |f| \le 1, \ \operatorname{spt}(f) \subset V \} \quad (A)$$

Let $A \subset \mathbb{R}^n$ and define:

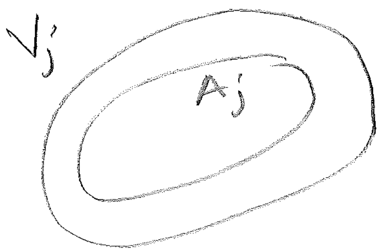
$$\mu(A) := \inf\{ \mu(V) \mid A \subset V \text{ open} \}. \quad (B)$$

Step 2: μ is an outer measure.

Let $\{A_j\}_{j=1}^{\infty}$ arbitrary sets and consider:

$$\bigcup_{j=1}^{\infty} A_j$$

Fix $\varepsilon > 0$. Choose V_j as follows:



$A_j \subset V_j$

$$\boxed{\mu(V_j) \le \mu(A_j) + \frac{\varepsilon}{2^j}} \quad (C)$$

From the definition (B):

$$A \subset B \implies \mu(A) \le \mu(B)$$

Hence

$$\boxed{\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \le \mu\left(\bigcup_{j=1}^{\infty} V_j\right)} \quad (D)$$

Let $V = \bigcup_{j=1}^{\infty} V_j$

$\boxed{\text{Let } g \in C_c(\mathbb{R}^n, \mathbb{R}^m),\ |g| \le 1,\ spt(g) \subset V}$

Let $K = spt(g) \subset V$

Let $\{\varphi_i\}$ a partition of unity subordinate to V. Then: (a) $0 \le \varphi_i \le 1$, (b) For each i, $\exists j(i)$ s.t. $spt(\varphi_i) \subset V_{j(i)}$ and (c) $\exists m$ s.t. $\varphi_1(x) + \varphi_2(x) + \cdots + \varphi_m(x) = 1$ in a neighborhood of K.

Then $g = \sum_{i=1}^{m} g\varphi_i$, since

$x \notin K \implies g(x) = 0 \implies \sum_{i=1}^{m} g(x)\varphi_i(x) = 0.$

$x \in K \implies g(x) = \sum_{i=1}^{m} g(x)\varphi_i(x) = g(x) \sum_{i=}^{m} \varphi_i(x) = g(x)$

Using Definition (A) and since $spt(g\varphi_i) \subset V_{j(i)}$:

and $$|L(g)| = \left|\sum_{i=1}^{m} L(g\varphi_i)\right| \le \sum_{i=1}^{m} |L(g\varphi_i)|$$

$$\le \sum_{i=1}^{m} \mu(V_{j(i)}) \le \sum_{j=1}^{\infty} \mu(V_j)$$

Taking sup over all g:

$$\boxed{\mu(V) = \mu\left(\bigcup_{j=1}^{\infty} V_j\right) \le \sum_{j=1}^{\infty} \mu(V_j)} \quad (E)$$

(C) + (D) + (E) $\Rightarrow$

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \le \mu\left(\bigcup_{j=1}^{\infty} V_j\right)$$

$$\le \sum_{j=1}^{\infty} \mu(V_j)$$

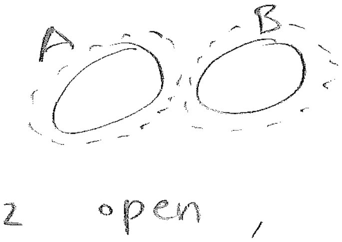
$$\le \sum_{j=1}^{\infty} \mu(A_j) + \frac{\varepsilon}{2^j}$$

$$= \sum_{j=1}^{\infty} \mu(A_j) + \varepsilon$$

$$\therefore \boxed{\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \mu(A_j)}$$

<u>Step 3</u>: μ is a <u>Radon</u> measure (i.e, μ is <u>Borel</u> <u>regular</u> and <u>$\mu(K) < \infty$</u>, for every compact set $K \subset \mathbb{R}^n$).

We first show that μ is a Caratheodory outer measure. Let $A, B \subset \mathbb{R}^n$, $d(A,B) > 0$.



Let U_1, U_2 open, $A \subset U_1$, $B \subset U_2$ with $d(U_1, U_2) > 0$.

Clearly,

$$\mu(U_1 \cup U_2) \leq \mu(U_1) + \mu(U_2)$$

We need to show:

$$\mu(U_1) + \mu(U_2) \leq \mu(U_1 \cup U_2)$$

$$\mu(U_2) = \sup\{L(g) : |g| \leq 1,\ \mathrm{spt}(g) \subset U_2\}$$

$$\mu(U_1) = \sup\{L(g) : |g| \leq 1,\ \mathrm{spt}(g) \subset U_1\}$$

$\exists\, g_1, g_2$ such that

$$\mu(U_1) - \varepsilon \leq L(g_1) \leq \mu(U_1), \quad \mathrm{spt}(g_1) \subset U_1$$

$$\mu(U_2) - \varepsilon \leq L(g_2) \leq \mu(U_2), \quad \mathrm{spt}(g_2) \subset U_2$$

$$\mu(U_1) + \mu(U_2) \leq L(g_1) + \varepsilon + L(g_2) + \varepsilon$$

$$= L(g_1 + g_2) + 2\varepsilon.$$

$$\mathrm{spt}(g_1+g_2) \subset U_1 \cup U_2 \Rightarrow L(g_1+g_2) \leq \sup\{L(g) : |g| \leq 1,\ \mathrm{spt}(g) \subset U_1 \cup U_2\} + 2\varepsilon$$

$$= \mu(U_1 \cup U_2) + 2\varepsilon$$

$$\therefore\ \mu(U_1) + \mu(U_2) \leq \mu(U_1 \cup U_2) + 2\varepsilon.$$

$$\varepsilon \to 0 \Rightarrow \mu(U_1) + \mu(U_2) \leq \mu(U_1 \cup U_2)$$

$$\therefore\ \mu(U_1) + \mu(U_2) = \mu(U_1 \cup U_2).$$

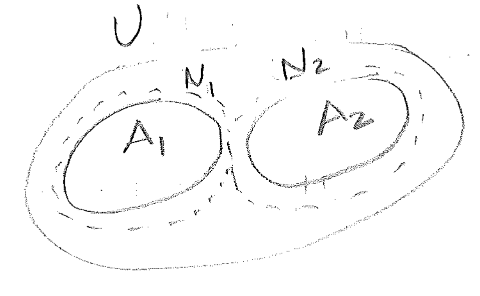
Clearly, $\mu(A_1 \cup A_2) \leq \mu(A_1) \cup \mu(A_2)$

8.83

By definition (B):

$\exists U$ open $A_1 \cup A_2 \subset U$ s.t.

$$\mu(A_1 \cup A_2) \leq \mu(U) \leq \mu(A_1 \cup A_2) + \varepsilon$$



N_1, N_2 open

Let

$U_1 := N_1 \cap U$ open

$U_2 = N_2 \cap U$ open.

$$\mu(A_1) + \mu(A_2) \leq \mu(U_1) + \mu(U_2);$$ by (B) and using $A \subset U_1$, $A \subset U_2$

previous step $= \mu(U_1 \cup U_2)$

$\leq \mu(U)$; $U_1 \cup U_2 \subset U$

$\leq \mu(A_1 \cup A_2) + \varepsilon$

$\varepsilon \to 0$ yields

$$\mu(A_1) + \mu(A_2) \leq \mu(A_1 \cup A_2).$$

$\therefore$ $\boxed{\mu(A_1) + \mu(A_2) = \mu(A_1 \cup A_2)}$

$\therefore$ μ is a Caratheodory outer measure

$\therefore$ Closed sets are μ-measurables

$\therefore$ Borel sets are μ-measurables

$\therefore$ $\boxed{\mu \text{ is a Borel measure}}$

Claim: μ is regular.

Let $A \subset \mathbb{R}^n$

$\Rightarrow \exists V_K,\ A \subset V_K$ open s.t

$$\mu(V_K) \leq \mu(A) + \frac{1}{K}, \quad \forall K.$$

Define:

$$G = \bigcap_{K=1}^{\infty} V_K$$

$\therefore$ $A \subset G$ and clearly

$$\mu(A) \leq \mu(G).$$

Also:

$$\mu(G) \leq \mu(V_K) \quad ; \text{ since } G \subset V_K, \forall K$$

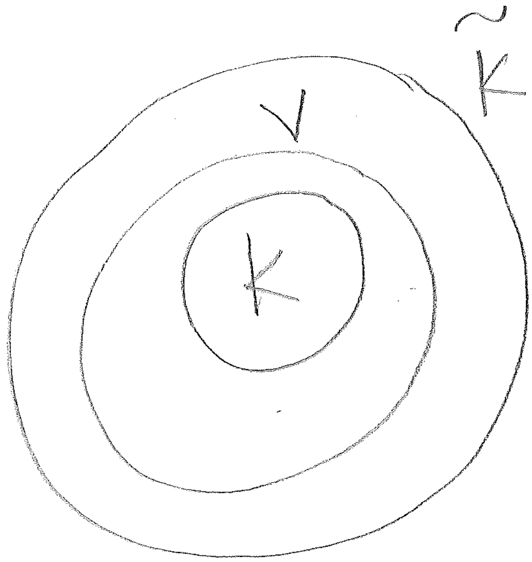
$$\leq \mu(A) + \frac{1}{K}$$

$K \to \infty$ yields:

$$\mu(G) \leq \mu(A)$$

$$\therefore \boxed{\mu(A) = \mu(G), \quad G \text{ Borel}}$$

Claim: $\mu(K) < \infty$ if K is compact.

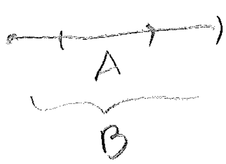


$K \subset V \subset \overline{V} \subset \tilde{K}$, V open

$K, \tilde{K}$ compact

$$\mu(K) \leq \mu(V) \quad ; \text{ by (B)}$$

$$= \sup\{L(g) : |g| \leq 1, \ \mathrm{Spt}(g) \subset V\}$$



$$\leq \sup\{L(g) : |g| \leq 1, \ \mathrm{Spt}(g) \subset \tilde{K}\}$$

$$< \infty \quad ; \text{ by } (*).$$

$\therefore \quad \mu(K) < \infty \quad \forall K$ compact

We conclude:

μ is a Radon outer measure.