Step 7: $\exists\, \sigma : \mathbb{R}^n \to \mathbb{R}^m$ μ-measurable such that

$$L(f) = \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu \quad , \quad \forall f \in C_c(\mathbb{R}^n; \mathbb{R}^m).$$

We have proved that:

$$\lambda(f) = \sup\{ |L(g)| : g \in C_c(\mathbb{R}^n; \mathbb{R}^m), \ |g| \le f \}$$

Can be represented as:

$$\boxed{\lambda(f) = \int_{\mathbb{R}^n} f \, d\mu , \quad \forall f \in C_c^+(\mathbb{R}^n)}$$

Fix $e \in \mathbb{R}^m$, $|e| = 1$. Define:

$$\lambda_e(f) := L(fe) , \quad f \in C_c(\mathbb{R}^n)$$

λ_e is linear, and

$$\begin{aligned} |\lambda_e(f)| &= |L(fe)| \\ &\le \sup\{ |L(g)| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m), \ |g| \le |f| \} \\ &= \lambda |f| \\ &= \int_{\mathbb{R}^n} |f| \, d\mu \end{aligned}$$

$\therefore$

$$|\lambda_e(f)| \leq \|f\|_{1,\mu} \quad \forall f \in C_c(\mathbb{R}^n)$$ (8.94)

Since:

<u>$C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n;\mu)$</u>

then:

λ_e can be extended to a bounded linear functional on $L^1(\mathbb{R}^n,\mu)$

$\therefore$ Since $\left(L^1(\mathbb{R}^n,\mu)\right)^* \simeq L^\infty(\mathbb{R}^n,\mu)$,

$\exists\, \sigma_e \in L^\infty(\mu)$ s.t.

(A) $\boxed{\lambda_e(f) = \int_{\mathbb{R}^n} f\,\sigma_e\,d\mu}$, $f \in C_c(\mathbb{R}^n)$.

Define:

$$\sigma = \sum_{j=1}^{m} \sigma_{e_j} e_j, \qquad \{e_1, e_2, \dots, e_m\} \text{ base of } \mathbb{R}^m.$$

$$= (\sigma_{e_1}, \sigma_{e_2}, \dots, \sigma_{e_m}) \qquad e_i = (0, \dots, 0, 1, 0, \dots 0).$$

(8.95)

For $f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$.

$$f = (f \cdot e_1)e_1 + (f \cdot e_2)e_2 + \cdots + (f \cdot e_m)e_m$$

$$\begin{aligned}
L(f) &= L\big((f \cdot e_1)e_1 + \cdots + (f \cdot e_m)e_m\big) \\
&= L((f \cdot e_1)e_1) + \cdots + L((f \cdot e_m)e_m) \\
&= \lambda_{e_1}(f \cdot e_1) + \cdots + \lambda_{e_m}(f \cdot e_m) \\
&= \int_{\mathbb{R}^n} (f \cdot e_1)\sigma_{e_1}\, d\mu + \cdots + \int_{\mathbb{R}^n} (f \cdot e_m)\sigma_{e_m}\, d\mu \\
&= \int_{\mathbb{R}^n} f \cdot (\sigma_{e_1}, \sigma_{e_2}, \ldots, \sigma_{e_m})\, d\mu \\
&= \int_{\mathbb{R}^n} f \cdot \sigma\, d\mu
\end{aligned}$$

$$\therefore \boxed{L(f) = \int_{\mathbb{R}^n} f \cdot \sigma\, d\mu, \quad f \in C_c(\mathbb{R}^n, \mathbb{R}^m).}$$

Step 8 $|\sigma| = 1$ μ-a.e.

We will use.

Thm (Extension of continuous functions: Measure theory and fine properties of functions, section 1.2).

Suppose $K \subset \mathbb{R}^n$ is compact and $f: K \to \mathbb{R}^m$ is continuous. There exists a continuous mapping $\bar{f}: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\bar{f} = f \text{ on } K.$$

Lusin's Theorem: Let μ be a Borel regular measure on $\mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^m$ be μ-measurable. Assume $A \subset \mathbb{R}^n$ is μ-measurable and $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then there exists a compact set $K \subset A$ such that:

(i) $\mu(A \setminus K) < \varepsilon$, and

(ii) $f|_K$ is continuous.

Lebesgue Points

Thm: Let μ be a Radon measure on $\mathbb{R}^n$ (i.e, μ is Borel, regular, $\mu(K) < \infty$, K compact) and $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu = f(x) \text{ for}$$

μ-a.e. $x \in \mathbb{R}^n$.

Lemma: We have $\sigma \in L^\infty(\mathbb{R}^n, \mu)$, (8.97)
$\sigma : \mathbb{R}^n \to \mathbb{R}^m$. Let $U \subset \mathbb{R}^n$ open.

Then:

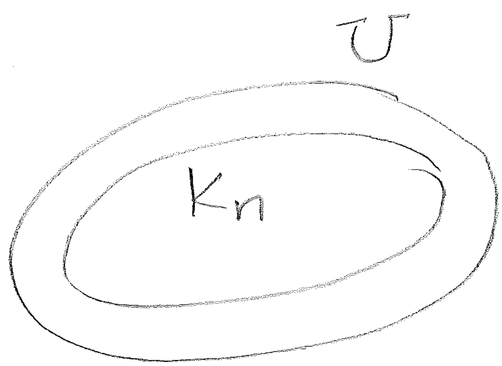
$\exists \{f_n\}$, $f_n \in C_c(\mathbb{R}^n; \mathbb{R}^m)$, $|f_n| \leq 1$, $\mathrm{Spt}(f_n) \subset U$

and

$$f_n \to \frac{\sigma}{|\sigma|} \quad \mu\text{-a.e. in } U$$

Note: Given $U \subset \mathbb{R}^n$ open, $|\sigma| > 0$
μ-almost everywhere on U

Proof:



Lusin's Theorem $\Rightarrow$
$\forall n$, $\exists K_n \subset U$, K_n compact such that

$$\frac{\sigma}{|\sigma|}\Big|_{K_n} \text{ is continuous}$$

$$\mu(U \setminus K_n) < \frac{1}{2^n}$$

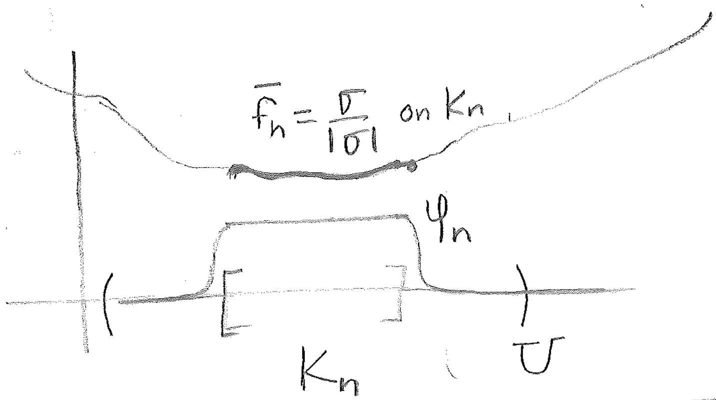
Extension Theorem $\Rightarrow$
$\exists \bar{f}_n : \mathbb{R}^n \to \mathbb{R}^m$ continuous s.t

$$\bar{f}_n \equiv \frac{\sigma}{|\sigma|} \text{ on } K_n.$$

Define:

$\boxed{f_n := \bar{f}_n \varphi_n}$, where φ is an smooth function with

$0 \leq \varphi_n \leq 1$, $\varphi_n \equiv 1$ on K_n; $\operatorname{spt}(\varphi_n) \subset U$



and such that $\boxed{|f_n| \leq 1, \ \operatorname{spt}(f_n) \subset U.}$

Define:

$$E_n := U \setminus K_n.$$

$$E := \bigcap_{K=1}^{\infty} \bigcup_{n=K}^{\infty} E_n.$$

Note that $\mu(E) = 0$ Since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$,

If $x \notin E$ then $\exists K_0$ s.t $x \notin E_n$ $\forall n \geq K_0$.

Hence,

$$f_n(x) = \frac{\sigma(x)}{|\sigma(x)|} \quad \forall n \geq K_0.$$

This implies that:

$$f_n(x) \longrightarrow \frac{\sigma(x)}{|\sigma(x)|} \quad \forall x \notin E.$$

$\therefore$ $f_n \rightarrow \frac{\sigma}{|\sigma|}$, μ-almost everywhere. ☒

We are now ready to show that:

$$|\sigma| = 1 \quad \mu\text{-a.e.}$$

Let $U \subset \mathbb{R}^n$ open, $\mu(U) < \infty$. From Lemma,
Since:

$$f_n \to \frac{\sigma}{|\sigma|} \quad \mu\text{-a.e}$$

then

$$f_n \cdot \sigma \to \frac{\sigma \cdot \sigma}{|\sigma|} \quad \mu\text{-a.e}$$

$$\therefore \quad \boxed{f_n \cdot \sigma \to |\sigma| \quad \mu\text{-a.e.}}$$

Since $|f_n| \leq 1$, $\mathrm{Spt}(f_n) \subset U$, and $\sigma = (\sigma_{e_1}, \ldots, \sigma_{e_m})$, $|\sigma_{e_i}| \leq 1$, the Lebesgue Dominated Convergence Theorem gives:

$$\int_U |\sigma| \, d\mu = \lim_{n\to\infty} \int_U f_n \cdot \sigma$$

$$\leq \mu(U) \; ; \quad \text{by definition of } \mu(U):$$

$\mu(U) := \sup\{L(f) : f \in C_c(\mathbb{R}^n, \mathbb{R}^m) : |f| \leq 1, \mathrm{Spt}(f) \subset U\} = \sup\{\int_{\mathbb{R}^n} f_n \cdot \sigma : |f| \leq 1, \mathrm{Spt}(f) \subset U\}$

On the other hand:

$$\int_U f \cdot \sigma \, d\mu \leq \int_U |\sigma| \, d\mu, \quad \forall f \in C_c(\mathbb{R}^n, \mathbb{R}^m), \; |f| \leq 1, \; \mathrm{Spt}(f) \subset U.$$

Taking sup over all such f:

$$\mu(U) \leq \int_U |\sigma| \, d\mu.$$

Hence

$$\boxed{\mu(U) = \int_U |\sigma| \, d\mu}$$

$\forall \, U \subset \mathbb{R}^n$ open, $\mu(U) < \infty$.

Replacing U with balls:

$$\frac{\int_{B(x,r)} |\sigma| \, d\mu}{\mu(B(x,r))} = 1$$

Taking $r \to 0$ and using the Theorem for Lebesgue points we conclude:

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |\sigma| \, d\mu = |\sigma(x)| = 1,$$

for μ-a.e. x.

Thus:

$$|\sigma(x)| = 1, \quad \mu\text{-a.e. } x.$$

Appendix for RRT3:

Slicing lemma:

Let $f \in C_c^+(\mathbb{R}^n)$. Let $K = \mathrm{spt}(f)$ and let μ be a Radon measure in $\mathbb{R}^n$. Then,

$$\mu(f^{-1}(t)) = 0,$$

except for (at most) a countable set $\{f^{-1}(t_i)\}$, for which we have

$$\mu(f^{-1}(t_i)) > 0.$$

Proof:

Define

$$L_K = \{ 0 < t \leq \|f\|_\infty : \mu(f^{-1}(t)) \geq \tfrac{1}{K} \}$$

Suppose that L_K is an infinite set. Since every infinite set contains a countable subset, it follows that there exists:

$$s_1 < s_2 < s_3, \dots \text{ with } s_i \in L_K.$$

Then

$$\mu\left(\bigcup_{i=1}^{\infty} f^{-1}(s_i)\right) = \sum_{i=1}^{\infty} \mu(f^{-1}(s_i)); \quad \text{since } f^{-1}(s_i) \cap f^{-1}(s_j) = \emptyset, \; s_i \neq s_j.$$

$$\geq \sum_{i=1}^{\infty} \frac{1}{k} = \infty. \qquad (B).$$

On the other hand:

$$\mu\left(\bigcup_{i=1}^{\infty} f^{-1}(s_i)\right) \leq \mu(K); \quad \text{since } \bigcup_{i=1}^{\infty} f^{-1}(s_i) \subset K$$

$$< \infty; \quad \text{since } \mu \text{ is Radon};$$

but this contradicts (B).

We conclude that:

L_k is a finite set.

Let:

$$L = \{ 0 < t \leq \|f\|_\infty : \mu(f^{-1}(t)) > 0 \},$$

Note that:

$$L = \bigcup_{k=1}^{\infty} L_k.$$

Since each L_k is finite, then L is a countable set, say $t_1, t_2, t_3, \ldots$.

We conclude that $\mu(f^{-1}(t)) = 0$ except for (at most) the countable set $\{\mu(f^{-1}(t_i))\}$, for which $\mu(f^{-1}(t_i)) > 0$.