Distributions

Let $\Omega \subset \mathbb{R}^n$ open set.
In previous chapter we were working with the space:

$$C_c(\Omega) = \{ f : \Omega \to \mathbb{R},\ f \text{ continuous},\ \operatorname{spt}(f) \subset \Omega \}$$

For $1 \leq K \leq \infty$ we define:

$$C^K(\Omega) = \{ f : \Omega \to \mathbb{R} : \text{the partial derivatives of } f \text{ of all orders (up-to and including } K) \text{ are continuous} \}$$

$$C_c^K(\Omega) = \{ f \in C^K(\Omega) : \operatorname{spt}(f) \subset \Omega \}.$$

We define:

$$\boxed{\mathcal{D}(\Omega) = C_c^\infty(\Omega)}$$

In previous chapter we consider certain linear functionals L defined on $C_c(\mathbb{R}^n)$ and we identify them with measures

$$L : C_c(\mathbb{R}^n) \to \mathbb{R} \longleftrightarrow \mu$$

We want to consider now linear functionals T defined on $\mathcal{D}$.

Since:

$\mathcal{D}(\Omega)$ is smaller than $C_c(\Omega)$,

then these functionals I will be more general than measures.

We now want to prove that:

$\exists F : \mathbb{R} \to \mathbb{R}$, such that F is infinitely differentiable and $F \equiv 0$ of $|x| \geq 1$;

that is, $F \in \mathcal{D}(\mathbb{R})$.

Define;

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Observe that f is C^∞ on $\mathbb{R} \setminus \{0\}$.

It remains to show that all derivatives exist and are continuous. at $x = 0$.

We have

$$f'(x) = \begin{cases} \frac{1}{x^2} e^{-1/x}, & x > 0 \\ 0, & x < 0. \end{cases}$$

and

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{-1/x} = \lim_{x \to 0^+} \frac{1}{e^{1/x}} = \lim_{\theta \to \infty} \frac{1}{e^\theta} = 0 = f(0)$$

$\therefore \lim_{x \to 0} f(x) = 0$, hence f is continuous at $x = 0$.

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Claim: f' exists and is continous at $x=0$.

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h}, \quad h>0.$$

$$\lim_{h\to 0^+} \frac{e^{-1/h}}{h}$$

$$= \lim_{\theta\to\infty} \theta e^{-\theta}, \quad \theta = \frac{1}{h}$$

$$= \lim_{\theta\to\infty} \frac{\theta}{e^{\theta}} = \lim_{\theta\to\infty} \frac{1}{e^{\theta}} = 0.$$

$\therefore \lim_{h\to 0} \frac{f(h)-f(0)}{h} = f'(0) = 0 \Rightarrow f'$ exists at $x=0$

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{1}{x^2} e^{-1/x}$$

$$= \lim_{\theta\to\infty} \theta^2 e^{-\theta}, \quad \frac{1}{x} = \theta$$

$$= \lim_{\theta\to\infty} \frac{\theta^2}{e^{\theta}}$$

$$= \lim_{\theta\to\infty} \frac{2\theta}{e^{\theta}}$$

$$= \lim_{\theta\to\infty} \frac{2}{e^{\theta}} = 0.$$

$\therefore \lim_{x\to 0} f'(x) = 0 = f'(0)$

$\therefore f'$ is continuous at $x=0$.

Claim: f'' exists and is continuous at $x=0$.

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We have $f'(x) = \begin{cases} \frac{1}{x^2} e^{-1/x}, & x>0 \\ 0, & x \leq 0 \end{cases}$

$$f''(x) = \frac{1}{x^2} \cdot \frac{1}{x^2} e^{-1/x} + e^{-1/x}\left(\frac{-2}{x^3}\right)$$

$$= \frac{1}{x^4} e^{-1/x} - \frac{2}{x^3} e^{-1/x}$$

$$= \left(\frac{1}{x^4} - \frac{2}{x^3}\right) e^{-1/x}, \quad x>0.$$

$f''(x) = 0, \quad x<0.$

What happen if $x=0$?

$$\lim_{h\to 0^+} \frac{f'(h)-f'(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{h^2} e^{-1/h} - 0}{h} = \lim_{h\to 0} \frac{1}{h^3} e^{-1/h}$$

$$= \lim_{\theta\to\infty} \theta^3 e^{-\theta} = \lim_{\theta\to\infty} \frac{\theta^3}{e^\theta}$$

$$= \lim_{\theta\to\infty} \frac{3\theta^2}{e^\theta} = \lim_{\theta\to\infty} \frac{6\theta}{e^\theta} = \lim_{\theta\to\infty} \frac{6}{e^\theta} = 0$$

$$\therefore \lim_{h\to 0} \frac{f'(h)-f'(0)}{h} = 0 = f''(0)$$

Also:

$$\lim_{x \to 0^+} f''(x) =$$

$$= \lim_{x \to 0^+} \frac{1}{x^2} e^{-1/x} = 0$$

$$\therefore \lim_{x \to 0} f''(x) = 0 = f''(0)$$

$\therefore$ f'' is continuous at $x=0$.

Clearly, we can continue this process to conclude that

$$\boxed{f^{(K)}(0) = 0 \quad \forall K = 1, 2, \ldots}$$

Define now:

$$F(x) = f(1 - |x|^2)$$

Note that $1 - |x|^2 = 1 - x_1^2 - \cdots - x_n^2$ which is a polynomial (infitely differentiable).

$$\therefore \boxed{\begin{array}{l} F \text{ is infinitely differentiable} \\ F \geq 0, \quad F \equiv 0 \text{ for } |x| \geq 1. \end{array}}$$