Recall, with

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

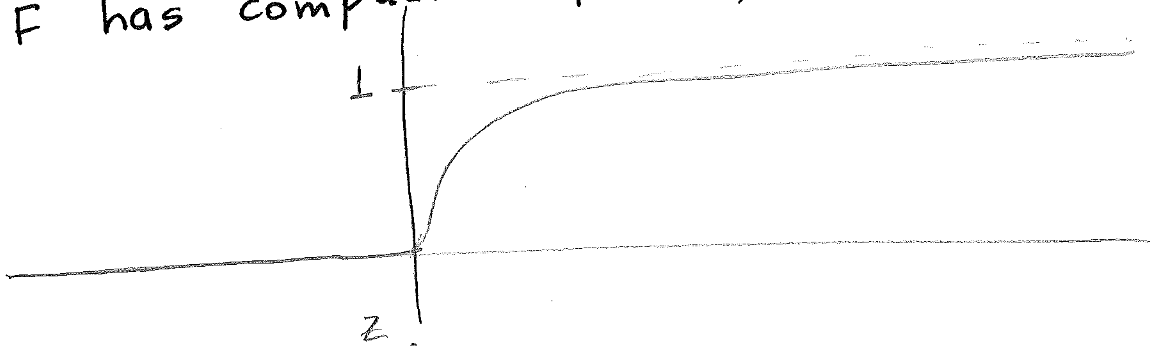
we can form:

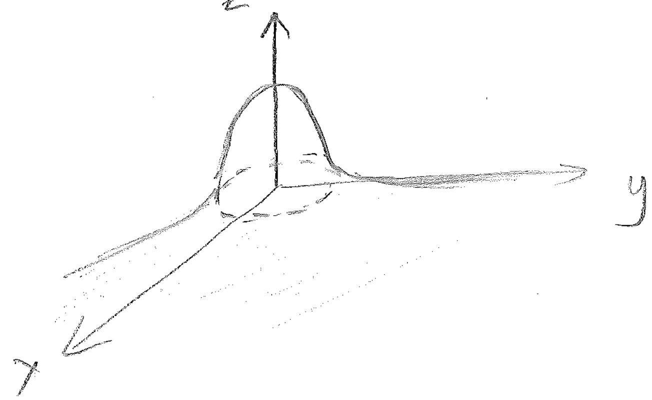
$$F : \mathbb{R}^n \to \mathbb{R}$$

$$F(x) = f(1 - |x|^2)$$

$$\therefore \quad \boxed{F(x) = \begin{cases} e^{\frac{1}{|x|^2 - 1}}, & |x| < 1 \\ 0, & |x| \ge 1. \end{cases}}$$

We have shown that $F \in C^\infty$, and since F has compact support, $F \in \mathcal{D}(\mathbb{R}^n)$.





<u>Note</u>: It is traditional in many parts of analysis to denote C^∞ functions with compact support by φ.

Let $\underline{\varphi \in C_c^\infty(\mathbb{R}^n)}$, $\underline{\varphi = 0 \text{ for } |x| \geq 1.}$

By multiplying φ by a suitable constant we can assume:

$$\boxed{\int_{B(0,1)} \varphi(x)\, d\lambda(x) = 1}$$

Define:
$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right)$$

Note:
$\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, $\varphi = 0$ for $|x| \geq \varepsilon$, $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)\, d\lambda(x) = 1$.

<u>Note</u>: The functions in $\mathcal{D}(\Omega)$ are often called <u>test functions</u>.

Def: Let $f \in L^1_{loc}(\mathbb{R}^n)$. Let

$$f_\varepsilon := f * \varphi_\varepsilon(x) = \int_{\mathbb{R}^n} f(x-y)\varphi_\varepsilon(y)\, d\lambda(y)$$

denote the mollifier of f.

Thm:
(i) If $f \in L^1_{loc}(\mathbb{R}^n)$, then $f_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\forall \varepsilon > 0$

(ii) $\lim_{\varepsilon \to 0} f_\varepsilon(x) = f(x)$, for every Lebesgue point x. That is, $f_\varepsilon \to f$ pointwise λ- almost everywhere.

(iii) If f is continuous, then
$f_\varepsilon \to f$ uniformly
on every compact set $K \subset \mathbb{R}^n$.

(iv) If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, then
$f_\varepsilon \in L^p(\mathbb{R}^n)$, $\|f_\varepsilon\|_p \le \|f\|_p$ and

$$\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_p = 0.$$

Proof

(i) We will first show:

$$\frac{\partial}{\partial x_i}(\varphi_\varepsilon * f) = \left(\frac{\partial \varphi_\varepsilon}{\partial x_i}\right) * f, \quad i = 1, 2, \dots, n$$

Fix $i \in \{1, 2, \dots, n\}$, $e_i = (0, 0, \dots, \underbrace{1}_{i\text{th position}}, 0, \dots 0)$

$$f_\varepsilon(x + he_i) - f_\varepsilon(x) = \int_{\mathbb{R}^n} [\varphi_\varepsilon(x + he_i - z) - \varphi_\varepsilon(x - z)] f(z)\, d\lambda(z)$$

Define:

$$\boxed{\alpha(t) = f(z)\, \varphi_\varepsilon(x - z + te_i)}$$

$\alpha : \mathbb{R} \to \mathbb{R}$ is C^∞.

$$\Rightarrow \alpha'(t) = f(z) \Big[\frac{\partial \varphi_\varepsilon}{\partial x_1}(x - z + te_i) \cdot 0 + \frac{\partial \varphi_\varepsilon}{\partial x_2}(x - z + te_i) \cdot 0$$
$$+ \dots + \frac{\partial \varphi_\varepsilon}{\partial x_i}(x - z + te_i) \cdot 1 + \dots + \frac{\partial \varphi_\varepsilon}{\partial x_n}(x - z + te_i) \cdot 0 \Big]$$

[Note that $x - z + te_i = (x_1 - z_1, x_2 - z_2, \dots, \underbrace{x_i - z_i + t}_{i\text{-th position}}, \dots, x_n - z_n)$]

$$\therefore \alpha'(t) = f(z) \frac{\partial \varphi_\varepsilon}{\partial x_i}(x - z + te_i)$$

$\Rightarrow$

$$\boxed{\begin{aligned} f_\varepsilon(x + he_i) - f_\varepsilon(x) &= \int_{\mathbb{R}^n} [\alpha(h) - \alpha(0)]\, d\lambda(z) \\ &= \int_{\mathbb{R}^n} \left[\int_0^h \alpha'(t)\, dt\right] d\lambda(z) \end{aligned}}$$

∴

$$f_\varepsilon(x+he_i) - f_\varepsilon(x) = \int_{\mathbb{R}^n} \int_0^h \frac{\partial \varphi_\varepsilon}{\partial x_i}(x-z+te_i) f(z)\, dt\, d\lambda(z)$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \underbrace{\frac{\partial \varphi_\varepsilon}{\partial x_i}(x-z+te_i)\, \chi_{(0,h)}(t)\, f(z)}_{G(t,z)}\, dt\, d\lambda(z)$$

Since $G(t,z) \in L^1(\mathbb{R}^n \times \mathbb{R})$ we can use Fubini's Theorem (see Thm. 189.1) and interchange the order of integration in the above integral. Thus:

$$f_\varepsilon(x+he_i) - f_\varepsilon(x) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{\partial \varphi_\varepsilon}{\partial x_i}(x-z+te_i)\, \chi_{(0,h)}(t)\, f(z)\, d\lambda(z)\, dt$$

$$= \int_0^h \underbrace{\int_{\mathbb{R}^n} \frac{\partial \varphi_\varepsilon}{\partial x_i}(x-z+te_i) f(z)\, d\lambda(z)}_{F(t)}\, dt$$

$$\boxed{\lim_{h\to 0} \frac{f_\varepsilon(x+he_i) - f_\varepsilon(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_0^h F(t)\, dt} \quad (A)$$

<u>Claim 1:</u> $F(t)$ is continuous.

If we define $\beta(h) = \int_0^h F(t)\, dt$ we have, by the Fundamental Theorem of Calculus:

$$\beta'(0) = F(0)$$
$$= \int_{\mathbb{R}^n} \frac{\partial \varphi_\varepsilon}{\partial x_i}(x-z) f(z)\, d\lambda(z)$$

Also, by definition:

$$\beta'(0) = \lim_{h \to 0} \frac{\beta(h) - \beta(0)}{h}$$

$$= \lim_{h \to 0} \frac{\int_0^h F(t)\,dt - 0}{h}$$

$$= \frac{1}{h} \int_0^h F(t)\, dt$$

Hence:

$$\lim_{h \to 0} \frac{f_\varepsilon(x + he_i) - f_\varepsilon(x)}{h} = \beta'(0)$$
$$= F(0)$$
$$= \int_{\mathbb{R}^n} \frac{\partial \varphi_\varepsilon}{\partial x_i}(x-z) f(z)\, d\lambda(z)$$

This shows that $\frac{\partial f_\varepsilon}{\partial x_i}$ exists and:

$$\boxed{\frac{\partial f_\varepsilon}{\partial x_i}(x) = \frac{\partial}{\partial x_i}(\varphi_\varepsilon * f)(x) = \left(\frac{\partial \varphi_\varepsilon}{\partial x_i} * f\right)(x)}$$

We now prove Claim 1:

We need to prove that

$$F(t) = \int_{\mathbb{R}^n} \frac{\partial \varphi_\varepsilon}{\partial x_i}(x - z + te_i)\, f(z)\, d\lambda(z)$$

is continuous.

Fix t_0 and let $t_n \to t_0$. We need to show that:

$$F(t_n) \longrightarrow F(t_0)$$

But:

$$F(t_n) = \int_{\mathbb{R}^n} \frac{\partial \varphi_\varepsilon}{\partial x_i}(x - z + t_n e_i)\, f(z)\, d\lambda(z).$$

Since $\frac{\partial \varphi_\varepsilon}{\partial x_i}$ has compact support:

$$\left| \frac{\partial f_\varepsilon}{\partial x_i}(\xi) \right| \leq M \quad \forall \xi \in \mathbb{R}^n,$$

Since $f \in L^1_{loc}(\mathbb{R}^n)$ then:

$$\left| \frac{\partial \varphi_\varepsilon}{\partial x_i} f \right| \in L^1(\mathbb{R}^n). \longrightarrow (1)$$

Also:

$$\frac{\partial \varphi_\varepsilon}{\partial x_i}(x - z + t_n e_i) f(z) \to \frac{\partial \varphi_\varepsilon}{\partial x_i}(x - z + t_0 e_i) f(z),$$

for λ-a.e. z.

Thus, Dominated Convergence Theorem gives:

$$\lim_{n \to \infty} \int_{\mathbb{R}^n} \frac{\partial \varphi_\varepsilon}{\partial x_i}(x - z + t_n e_i) f(z)\, d\lambda(z) = \int_{\mathbb{R}^n} \frac{\partial \varphi_\varepsilon}{\partial x_i}(x - z + t_0 e_i) f(z)\, d\lambda(z)$$

$\therefore \lim_{n\to\infty} F(t_n) = F(t_0)$

$\therefore$ F is continuous at t_0. ∎

Recall that we have proved:

$$\boxed{\frac{\partial}{\partial x_i}(\varphi_\varepsilon * f) = \frac{\partial \varphi_\varepsilon}{\partial x_i} * f}$$

Since $\frac{\partial \varphi_\varepsilon}{\partial x_i}$ is smooth with compact support and $f \in L^1_{loc}(\mathbb{R}^n)$, a similar argument used in the proof of Claim 1 gives:

$\frac{\partial \varphi_\varepsilon}{\partial x_i} * f$ is continuous

$\therefore$ $\frac{\partial}{\partial x_i}(\varphi_\varepsilon * f)$ is continuous

$\therefore$ $\varphi_\varepsilon * f \in C^1$

Proceeding by induction we see that:

$$\frac{\partial^2}{\partial x_j \partial x_i}(\varphi_\varepsilon * f) = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}(\varphi_\varepsilon * f)\right) =$$

$$= \frac{\partial}{\partial x_j}\left(\frac{\partial \varphi_\varepsilon}{\partial x_i} * f\right) = \frac{\partial^2 \varphi_\varepsilon}{\partial x_j \partial x_i} * f$$

The same argument in Claim 1 shows that $\frac{\partial^2 \varphi_\varepsilon}{\partial x_j \partial x_i} * f$ is continuous and hence $\varphi_\varepsilon * f \in C^2$.

Notation: Let

$\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Then we denote:

$$D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

By induction, we conclude:

$$D^\alpha(\varphi_\varepsilon * f) = (D^\alpha \varphi_\varepsilon) * f$$

As before, since $D^\alpha \varphi_\varepsilon * f$ is continuous, then $D^\alpha(\varphi_\varepsilon * f)$ is continuous too.

We conclude that:

$$f_\varepsilon = \varphi_\varepsilon * f \in C^\infty(\mathbb{R}^n).$$

(ii) Observe that:

$|f_\varepsilon(x) - f(x)|$

$$= \left| \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) f(y)\, d\lambda - \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) f(x)\, d\lambda \right|$$

$$\leq \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) |f(y) - f(x)|\, d\lambda(y); \quad \text{since } \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y)\, d\lambda = 1$$

$$\leq \max_{\mathbb{R}^n} \varphi\, \varepsilon^{-n} \int_{B(x,\varepsilon)} |f(x) - f(y)|\, d\lambda(y)$$

But

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} |f(y) - f(x)|\, d\lambda(y) = 0, \text{ for}$$

λ-a.e. x.

$\therefore \lim_{\varepsilon \to 0} |f_\varepsilon(x) - f(x)| = 0$ for λ-a.e. x.

$\therefore$ $\boxed{f_\varepsilon(x) \to f(x) \text{ for } \lambda\text{-a.e. } x.}$

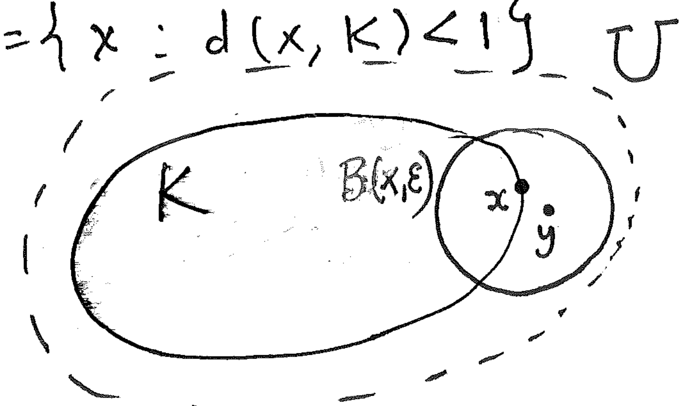
(iii) Consider the case:

f is continuous.

Fix $K \subset \mathbb{R}^n$ a compact set.

Let

$U = \{x : d(x, K) < 1\}$



Note:

f is uniformly continuous on K

f is uniformly continuous on $\overline{U}$

As in (ii) we have:

$$(*) \quad |f_\varepsilon(x) - f(x)| \le M \varepsilon^{-n} \int_{B(x,\varepsilon)} |f(y) - f(x)| \, d\lambda(y),$$

where $M = \sup_{\mathbb{R}^n} \varphi$.

Then:

$\forall \eta > 0 \quad \exists\, 0 < \varepsilon < 1$ s.t. $|f(x) - f(y)| < \eta$

From $(*)$, if $x \in K$:

$$|f_\varepsilon(x) - f(x)| \le M \cancel{\varepsilon^{-n}} \eta \cdot \omega(n) \cancel{\varepsilon^{n}} = M \omega(n) \eta.$$

η arbitrary $\Rightarrow$ $f_\varepsilon \to f$ uniformly on K.