Chapter 8

Elements of functional Analysis.

Review. L^p spaces.

<u>Def</u>: Let $1 \le p \le \infty$, $E \in \mathcal{M}$.

Let $L^p(E, \mathcal{M}, \mu)$. denote the class of all measurable functions f on E such that

$$\|f\|_{p,E;\mu} < \infty, \text{ where}$$

$$\|f\|_{p,E;\mu} := \begin{cases} \left(\int_E |f|^p \, d\mu\right)^{1/p}, & 1 \le p < \infty. \\ \inf\{M : |f| \le M \ \ \mu\text{-a.e. on } E\}, & p = \infty \end{cases}$$

$\|f\|_{p,E;\mu}$ is the L^p <u>norm</u> of f on E.

<u>Def</u>: Let X be a set and $\mathcal{M}$ a σ-algebra of subsets of X. A measure on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ s.t:

(i) $\mu(\emptyset) = 0$

(ii) If $\{E_i\}$ is a sequence of disjoints sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

The triple $(X, \mathcal{M}, \mu)$ is called a measure space and the sets in $\mathcal{M}$ are called measurable sets.

Notations :

In many applications we have:

$X = \mathbb{R}^n$, $\mathcal{M}$ = Set of Lebesgue measurable sets and μ is the Lebesgue measure. In the text book we use λ to denote Lebesgue measure; i.e.

$E \subset \mathbb{R}^n$ then $\lambda(E)$ is the Lebesgue measure of E.

Sometimes, we simply say.

- $\|f\|_p$ if the domain of f and the measure is clear from the context.
- $L^p(X)$ instead of $L^p(X, \mathcal{M}, \mu)$.

Def: A normed linear spaces is a vector space X with a norm.

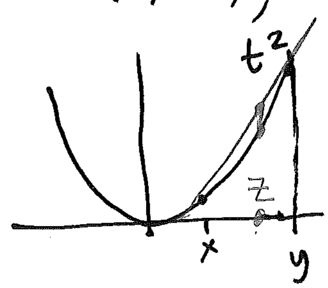
$L^P(X)$ is a <u>normed linear space</u>

because it is a vector space of functions with a norm. (we identify: "$f=g$" as $f=g$ μ-a.e.).

<u>Vector space</u>: • $f, g \in L^P(X) \Rightarrow f+g \in L^P(X)$

use $|a+b|^P \leq 2^{P-1}(|a|^P + |b|^P)$

$t \mapsto t^P$ is convex on $(0,\infty)$, $P \geq 1$, $P < \infty$



• $f \in L^P(X)$, $c \in \mathbb{R} \Rightarrow cf \in L^P(X)$

$$f(z) = f[\underbrace{(1-t)x + ty}_{z}] \leq (1-t)f(x) + t f(y)$$

$a, b > 0$, $t = 1/2$

$$\left(\frac{1}{2}a + \frac{1}{2}b\right)^P \leq \frac{1}{2}a^P + \frac{1}{2}b^P$$

$$(a+b)^P \leq \frac{2^P}{2}(a^P + b^P)$$

$$(a+b)^P \leq 2^{P-1}(a^P + b^P)$$

$$\int_X |f+g|^P \, d\mu \leq \int_X (|f(x)| + |g(x)|)^P d\mu \leq 2^{P-1}\int_X (|f(x)|^P + |g(x)|^P)\, d\mu$$

$$\le 2^{p-1}\left(\int_X |f|^p\,d\mu + \int_X |g|^p\,d\mu\right)$$

$$< \infty.$$

$\|f\|_p$ defines a norm:

(i) $\|f\|_p \ge 0 \quad \forall f \in L^p(X)$

(ii) $\|f\|_p = 0 \iff f = 0 \quad \mu$-a.e.

(Problem 6.4, $f \ge 0$. $\int_X f\,d\mu = 0 \Rightarrow f = 0\ \mu$-a.e.)

(iii) Triangle inequality:

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

This is Minkowski's inequality:

$$\|f+g\|_p^p = \int_X |f+g|^p\,d\mu = \int_X |f+g|^{p-1}|f+g|$$

$$\le \int_X |f+g|^{p-1}|f|\,d\mu + \int_X |f+g|^{p-1}|g|\,d\mu$$

(Holder's inequality $\int_X |fg|\,d\mu = \int_X |f||g|\,d\mu \le \|f\|_p \|g\|_{p'}$)

$$\le \left(\int_X (|f+g|^{p-1})^{p'}\right)^{1/p'}\left(\int_X |f|^p\,d\mu\right)^{1/p} + \left(\int_X (|f+g|^{p-1})^{p'}\right)^{1/p'}\left(\int_X |g|^p\right)^{1/p}$$

$$\frac{1}{p}+\frac{1}{p'}=1$$

$$1+\frac{p}{p'}=p \quad \Rightarrow \quad \frac{p}{p'}=p-1$$

$$(p-1)p'=\frac{p}{p'}\cdot p'=p.$$

$$=\left(\int_X |f+g|^p\right)^{1/p'}\|f\|_p+\left(\int_X |f+g|^p\right)^{1/p'}\|g\|_p$$

$$\therefore \|f+g\|_p^p \le \left(\|f+g\|_p^p\right)^{1/p'}\|f\|_p+\left(\|f+g\|_p^p\right)^{1/p'}\|g\|_p$$

$$=\|f+g\|_p^{p-1}\|f\|_p+\|f+g\|_p^{p-1}\|g\|_p.$$

$$\|f+g\|_p \le \|f\|_p+\|g\|_p. \quad , \text{ if } \|f+g\|_p \neq 0$$

(The assertion is clear if $\|f+g\|_p=0$).

<u>Remark</u>: For $1\le p\le\infty$ the spaces $L^p(X)$ are normed linear spaces provided we agree to identify functions that are equal μ-a.e. The norm $\|\cdot\|_p$ induces a metric on $L^p(X)$ if we define:

$$\rho(f,g):=\|f-g\|_p$$

for $f,g\in L^p(X)$ and agree with "$f=g$" as $f=g$ μ-a.e.

<u>Def</u>: A linear space (or vector space) is a set X endowed with two operations, addition and scalar multiplication, that satisfy the following conditions: for any $x, y, z \in X$, $\alpha, \beta \in \mathbb{R}$

(i) $x + y = y + x \in X$

(ii) $x + (y+z) = (x+y) + z$

(iii) $\exists\, 0 \in X$ s.t. $x + 0 = x \quad \forall x \in X$.

(iv). $\forall x \in X \quad \exists\, w \in X$ s.t. $x + w = 0$

(v) $\alpha x \in X$

(vi) $\alpha(\beta x) = (\alpha\beta) x$

(vii) $\alpha(x+y) = \alpha x + \alpha y$

(viii) $(\alpha + \beta) x = \alpha x + \beta x$

(ix) $1 x = x$.

<u>Def</u>: Let X be a linear space. A function $\|\cdot\| : X \to \mathbb{R}$ is a norm on X if

(i) $\|x+y\| \le \|x\| + \|y\| \quad \forall x, y \in X$.

(ii) $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall x \in X, \ \forall \alpha \in \mathbb{R}$

(iii) $\|x\| \geq 0 \quad \forall x \in X$ (iv) $\|x\| = 0 \iff x = 0$

<u>Def</u>: A linear space X equipped with a norm $\|\cdot\|$ is a normed linear space.

If X is a normed linear space with norm $\|\cdot\|$ then, for $x, y \in X$ set:

$$\rho(x, y) = \|x - y\|$$

Then ρ defines a metrix on X since

(i) $\rho(x,y)=0 \iff x-y$

(ii) $\rho(x,y) = \rho(y,x) \quad \forall x, y \in X$

(iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z) \quad \forall x, y, z \in X$

<u>Def</u>: A normed linear space X is a <u>Banach space</u> if it is a complete metric space with respect to the metric induced by its norm.

Recall that the metric space (X, ρ) is complete if every cauchy sequence $\{x_k\}$ in X converges ~~in X~~. to a point $x \in X$; i.e. $\forall \varepsilon > 0 \ \exists N$ s.t

$$\rho(x_k, x) < \varepsilon \quad \forall k \geq N.$$

Recall that $\{x_k\}$ Cauchy means:

$\forall \varepsilon > 0 \quad \exists N \quad$ s.t.

$$\rho(x_k, x_m) < \varepsilon \qquad \forall k, m \geq N.$$

Ex : • $\mathbb{R}^n$ is a Banach space with respect to the norm

$$\| (x_1, x_2, \dots, x_n) \| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

• For $1 \le p \le \infty$ the linear spaces $L^p(X, \mu)$ are Banach spaces with respect to the norms

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f|^p d\mu \right)^{1/p}, \quad 1 \le p < \infty$$

$$\|f\|_{L^p(X,\mu)} = \inf \{ M : \mu(\{x \in X : |f(x)| > M\}) = 0 \},$$
$$p = \infty$$

• If X is a compact metric space, then the linear space $C(X)$ of all continuous real-valued functions on X is a Banach space with respect to the norm

$$\|f\| = \sup_{x \in X} |f(x)|$$

$$C(X) = \{ f : X \to \mathbb{R},\ f \text{ continuous} \}$$ 8.10

Clearly $C(X)$ is a linear space

Def: $\|f\| = \sup_{x \in X} |f(x)|$

$\|\cdot\|$ is a norm in $C(X)$.

Let $\{f_k\} \subset C(X)$ a Cauchy sequence;

i.e. $\forall \varepsilon > 0 \quad \exists N$ s.t

$$\|f_k - f_m\| < \varepsilon \quad \forall k, m \geq N$$

$$\therefore \sup_{x \in X} |f_k(x) - f_m(x)| < \varepsilon \quad \forall k, m \geq N.$$

$\therefore \{f_k(x)\}$ is Cauchy in $\mathbb{R}$, for every x.

$$\therefore \quad f_k(x) \longrightarrow l(x)$$

Define:

$$f(x) = l(x).$$

$$\Rightarrow \quad |f_k(x) - f_m(x)| < \varepsilon \quad \forall k, m \geq N$$

Fix K and let $m \to \infty$.

$$|f_K(x) - \lim_{m \to \infty} f_m(x)| < \varepsilon$$

$$\Rightarrow \quad |f_K(x) - f(x)| < \varepsilon$$

Since K and x was arbitrary, we obtain:

$$|f_k(x) - f(x)| < \varepsilon \quad \forall k \geq N, \ \forall x \in X$$

$$\therefore \ \|f_K - f\| < \varepsilon \quad \forall k \geq N$$

or $$d(f_K, f) < \varepsilon \quad \forall k \geq N$$

that is, $f_K \to f$ in $C(X)$; which is equivalent to say that $f_K \to f$ uniformly. Since the limit of uniformly continuous function is also a continuous function we conclude that $f \in C(X)$.

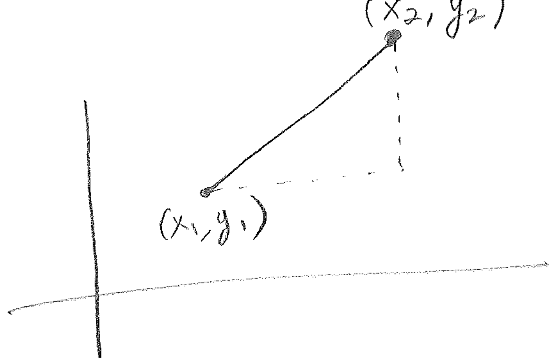
$\therefore$ $C(X)$ is a complete metric space (actually $C(X)$ is a Banach space).

<u>Ex</u>: We can have different metrics in an space X. Let $X = \mathbb{R}^2$.

$$\|(x, y)\|_2 = \sqrt{x^2 + y^2}$$

$$\rho((x_1, y_1), (x_2, y_2)) = \|(x_1 - x_2, y_1 - y_2)\| = \|(x_2 - x_1, y_2 - y_1)\|$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



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Def:

$\|(x,y)\|_\infty = \max\{|x|, |y|\}$, $\|(x,y)\|_p = (|x|^p + |y|^p)^{1/p}$

$\|\cdot\|_\infty$ defines a norm in $\mathbb{R}^2$

$\|\cdot\|_p$ defines a norm in $\mathbb{R}^2$.

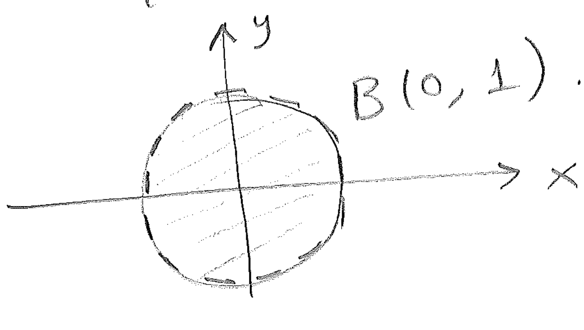
Ex: Draw the unit balls in $\mathbb{R}^2$ corresponding to the different metrics.

In a normed linear space X, we denote by $B(x,r)$ the open ball with center at x and radius r; ie,

$$B(x,r) = \{y \in X : \|y - x\| < r\}$$
$$= \{y \in X : \rho(y,x) < r\}$$

In $\mathbb{R}^2$; with $\|\cdot\|_2$, the unit ball $B(0,1)$ is:

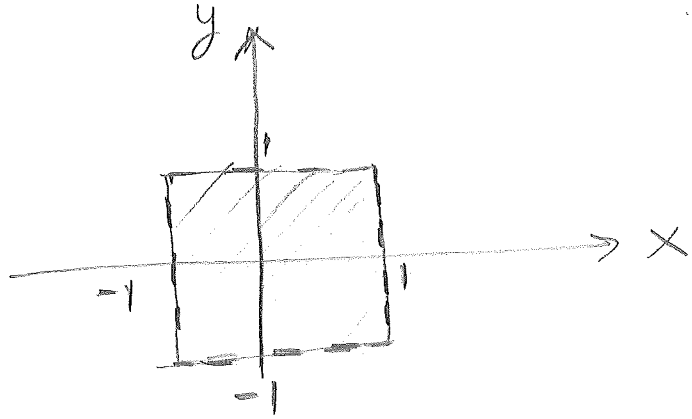
$$B(0,1) = \{(x,y) \in \mathbb{R}^2 : \|(x,y) - (0,0)\|_2 < 1\}$$
$$= \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\}$$



In $\mathbb{R}^2$, with $\|\cdot\|_\infty$,

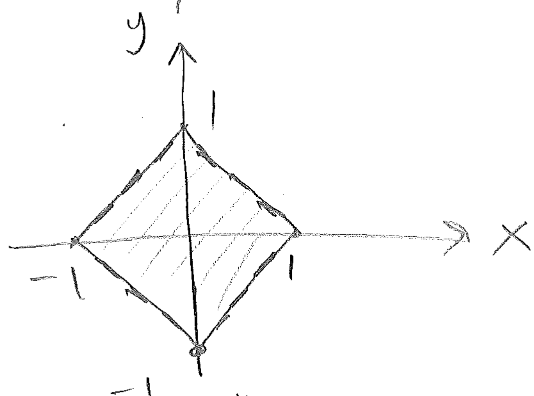
the unit ball $B(0,1)$ is:

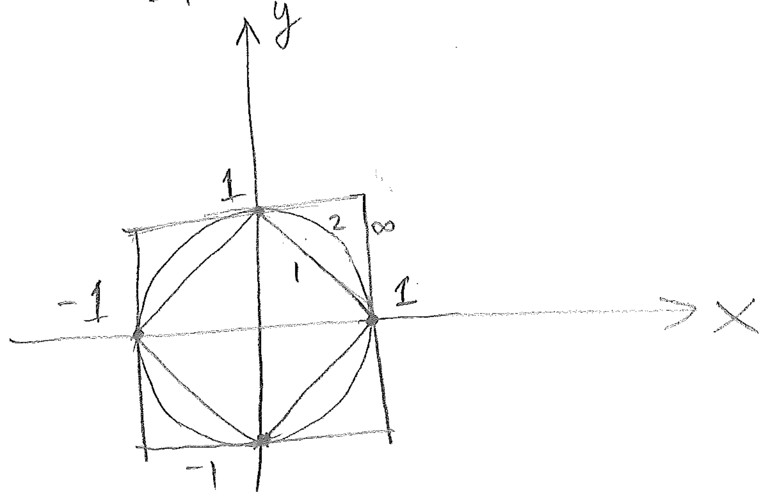
$$B(0,1) = \{(x,y) \in \mathbb{R}^2 : \|(x,y)-(0,0)\|_\infty < 1\}$$
$$= \{(x,y) \in \mathbb{R}^2 : \mathrm{Max}\{|x|, |y|\} < 1\}$$



With $\|\cdot\|_1$, the unit ball is:

$$B(0,1) = \{(x,y) \in \mathbb{R}^2 : |x| + |y| < 1\}$$





Def: Suppose X and Y are linear spaces. A mapping $T: X \to Y$ is linear if for any $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$:

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

Def: X, Y normed linear spaces and $T: X \to Y$ is a linear mapping, then T is bounded if there exists a constant M such that:

$$\|T(x)\| \leq M \|x\| \quad \forall x \in X.$$

Thm: X, Y normed linear spaces and $T: X \to Y$ is linear. Then

(i) T is bounded $\Leftrightarrow$ $\sup \{ \|T(x)\| : x \in X, \|x\| = 1 \} < \infty$

(ii) T is continuous $\Leftrightarrow$ T is bounded.

$\Longrightarrow$

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(i) Assume T bounded

Let $x \in X$, with $\|x\| = 1$.

$$\Rightarrow \|T(x)\| \leq M\|x\|$$

$$\Rightarrow \|T(x)\| \leq M, \quad \forall x \in X, \ \|x\| = 1$$

$$\therefore \sup\{\|T(x)\| : x \in X, \|x\| = 1\} \leq M < \infty$$

$\Longleftarrow$ Let $\alpha = \sup\{\|T(x)\| : x \in X, \|x\| = 1\}$

Let $x \in X$.

Consider $\frac{x}{\|x\|}$

Then:

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq \alpha.$$

$$\therefore \frac{1}{\|x\|}\|T(x)\| \leq \alpha$$

$$\therefore \|T(x)\| \leq \alpha\|x\|$$

$\therefore$ T is bounded

(ii) If T bounded, clearly T is continuous.

Recall $f : X \to Y$, X, Y metric spaces

f is continuos at $p \in X$ if $\forall \varepsilon > 0$, $\exists \delta$ s.t

$$\rho_X(x, p) < \delta \implies \rho_Y(f(x), f(p)) < \varepsilon.$$

If T is bounded, then we have that $\exists M$ s.t

$$\|T(x)\| \leq M\|x\| \quad \forall x \in X.$$

Then $\forall \varepsilon > 0$, $\|T(x)\| < \varepsilon \quad \forall x \in B(0, \frac{\varepsilon}{M})$.

Given $x_0 \in X$, we have

$$\|y - x_0\| < \frac{\varepsilon}{M} \implies \|T(y) - T(x_0)\| = \|T(y - y_0)\| < \varepsilon$$

Then, T is continuous on X.

Assume now that T is continuous.

In particular T is continuous at 0.

Then $\exists \delta > 0$ s.t.

$$x \in B(0, 2\delta) \implies \|T(x) - T(0)\| < 1$$

$T(0) = 0$ since T is linear.

If $x \in X$, $\|x\| = 1$, then

$$\|T(x)\| = \frac{1}{\delta}\|T(\delta x)\| \leq \frac{1}{\delta}; \quad \text{since } \|\delta x\| = \delta$$

and hence $\delta x \in B(0, 2\delta)$. We have shown

$$\sup\{\|T(x)\| : x \in X, \|x\| = 1\} < \infty$$

By (i) we conclude that T is bounded.

Def: Let X, Y normed linear spaces.

Define $\mathcal{B}(X,Y) = \{T : X \to Y : T \text{ linear and bounded}\}$

$\mathcal{L}(X,Y) = \{T : X \to Y : T \text{ linear}\}$

$\mathcal{B}(X,Y)$ is a subspace of $\mathcal{L}(X,Y)$

If $Y = \mathbb{R}$; then the elements of $\mathcal{L}(X,\mathbb{R})$ are called linear functionals on X.

Thm: Let X, Y normed linear spaces $T \in \mathcal{B}(X,Y)$. Set

$$(*) \quad \|T\| = \sup \{\|T(x)\| : x \in X, \|x\| = 1\}.$$

then $(*)$ defines a norm on $\mathcal{B}(X,Y)$. If Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space with respect to this norm.

8.4 Dual spaces.

Def. Let X be a normed linear space. The dual space X^* of X is the linear space of all bounded linear functionals on X eqquiped with the norm:

$$\|f\| = \sup \{ |f(x)| : x \in X, \ \|x\| = 1 \}.$$

Thus, with this definition we know that

$$X^* = \mathcal{B}(X, \mathbb{R}) \text{ is a Banach space.}$$

A metric space is separable if it contains a countable dense subset.

Three very important Theorems in Functional Analysis.

1.- The Hahn - Banach Theorem
2.- The open Mapping Theorem
3.- The Closed - Graph Theorem.

Hahn-Banach Theorem – norm version.

Suppose X is a normed linear space

Y is a subspace of X

$f : Y \to \mathbb{R}$ a linear functional, and

$|f(x)| \le M \|x\|$, $\forall x \in Y$,

Then:

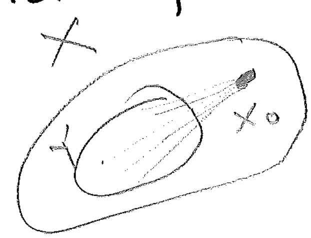
$\exists\, g : X \to \mathbb{R}$ continuous linear functional such that:

$g = f$ on Y and $|g(x)| \le M \|x\|$, $\forall x \in X$.

Corollary:

Suppose X is a normed linear space

Y is a subspace of X



$x_0 \in X$ and

$$\alpha = \inf_{y \in Y} \|y - x_0\| > 0 \quad (\text{i.e. } d(x_0, Y) > 0)$$

Then:

$\exists\; f : X \to \mathbb{R}$ bounded linear functional s.t:

$f \equiv 0$ on Y, $f(x_0) = 1$ and $\|f\| = \frac{1}{\alpha}$

The Open Mapping Theorem.

Let $T: X \to Y$ bounded, linear, on-to, X, Y Banach spaces

Then:

$T(U)$ is open in Y whenever U is an open subset of X.

Corollary:

Let $T: X \to Y$ be bounded, linear, 1-1 and on-to, X, Y Banach spaces

Then:

$T^{-1}: Y \to X$ is a bounded linear mapping.

The Closed Graph Theorem

Let $T: X \to Y$ linear, X, Y Banach spaces.

Assume that the graph of T, $\{(x, T(x)) : x \in X\}$ is closed in $X \times Y$

Then T is continuous

Thm: X is separable if X^* is separable.

Let $\{f_K\}$ be a countable dense subset of X^*.

For each K, $\exists x_K \in X$, $\|x_K\| = 1$ s.t.

$$|f_K(x_K)| \geq \frac{1}{2}\|f_K\|$$

Def:

$$W = \left\{ \sum a_K x_K, \quad a_K \in \mathbb{Q}; \text{ finite linear combination of elements of } \{x_K\} \right\}$$

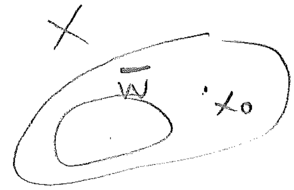
W is countable

If $\overline{W} = X$ then W is dense and we are done, ie, X is separable.

Assume $\overline{W} \subsetneq X$. Then $\exists x_0 \in X \setminus \overline{W}$.

and

$$d(x_0, \overline{W}) > 0$$



Hahn Banach $\Rightarrow$ $\exists f \in X^*$ s.t.

$$f \equiv 0 \text{ on } \overline{W} \quad f(x_0) = 1$$

$\{f_K\}$ dense in X^* $\Rightarrow$ $\exists \{f_{K_j}\}$ s.t.

$$\lim_{j \to \infty} \|f_{K_j} - f\| = 0$$

$\|x_{K_j}\| = 1 \Rightarrow \|f_{K_j} - f\| \geq \|f_{K_j}(x_{K_j}) - f(x_{K_j})\| = \|f_{K_j}(x_{K_j})\| \geq \frac{1}{2}\|f_{K_j}\|$

$\Rightarrow \|f_{K_j}\| \to 0 \quad \Rightarrow \|f\| - \|f_{K_j}\| \leq \|f_{K_j} - f\| \leq \varepsilon \quad \forall j \geq N(\varepsilon)$.

$\Rightarrow \|f\| \leq \|f_{K_j}\| + \varepsilon \quad \forall j \geq N(\varepsilon)$

Letting $j \to \infty$:

$\|f\| \leq 0 + \varepsilon$; since ε is arbitrary,

$\|f\| = 0$ i,e, $f \equiv 0$,

which contradicts $f(x_0) \neq 0$. We conclude

$$\overline{W} = X. \quad \square$$

<u>Def</u>: X, Y linear spaces.

If $T: X \to Y$ is one-to-one and on-to we say that T is a <u>linear isomorphism</u> and that X and Y are <u>isomorphic</u>.

If in addition, X and Y are normed linear spaces and $\|T(x)\| = \|x\| \ \forall x \in X$, then T is an <u>isometric isomorphism</u> and X and Y are <u>isometrically isomorphic</u>

<u>Notation</u>: X^{**} is the dual of X^*

<u>Def</u>:

$$\Phi : X \to X^{**} = (X^*)^*$$

$$\Phi(x)(f) = f(x)$$

Note:

$$|\Phi(x)(f)| = |f(x)| \leq \|f\| \, \|x\|.$$

$$\therefore \ |\Phi(x)(f)| \leq \|x\| \, \|f\|$$

$\therefore$ $\Phi(x)$ is a bounded linear functional defined on X^*; and moreover $\|\Phi(x)\| \leq \|x\|$

$\therefore$ $\Phi(x) \in X^{**}$ $\therefore$ Φ well defined

Φ bounded because:

$\|\Phi(x)\| \leq \|x\|$

and hence $\|\Phi\| \leq 1$

Φ is called the <u>natural imbedding</u> of X in X^{**}.

Φ is an isometric isomorphism of X onto $\Phi(X)$; since we will see in the next result that

$$\underset{(X^*)^*}{\|\Phi(x)\|} = \sup_{f \in X^*} \{ |\Phi(x)(f)| : \|f\| = 1 \}.$$

$$= \sup\{|f(x)| : f \in X^*, \|f\| = 1\}$$

$$= \|x\|.$$

$\therefore \|\Phi(x)\| = \|x\|, \forall x \in X.$

<u>Thm</u>: X normed linear space. Then, $\forall x \in X$:

$$\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| = 1\}$$

<u>Proof</u>: Let $\alpha = \sup\{|f(x)| : f \in X^*, \|f\| = 1\}$

$|f(x)| \leq \|f\| \|x\| \quad \forall x \in X$

$\leq \|x\| \quad$ if $\|f\| = 1$

$\therefore \alpha \leq \|x\|$

Let $x \neq 0$

$d(x, \{0\}) = \|x\| > 0$

$\Rightarrow \exists g \in X^*; \quad g(0) = 0$

$g(x) = 1$ and $\|g\| = \dfrac{1}{\|x\|}$

Def:

$f := \|x\| g \quad , \quad \|f\| = 1 \, , \quad f \in X^*$

$\Rightarrow |f(x)| = |\, \|x\| g(x) \,|$

$= \|x\|$

$\therefore \quad \alpha \geq \|x\|$

$\therefore \quad \alpha = \|x\|.$

Def: If $\Phi(X) = X^{**}$ then X is said to be reflexive. In this case

$$X \sim X^{**}$$

(X is isometrically isomorphic to X^{**})

Ex: $\mathbb{R}^n$ is reflexive

(ii) If $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$,

then:

$$\Psi : L^{p'}(X,\mu) \longrightarrow \underset{f \in L^p}{\left(L^p(X,\mu)\right)^*}$$

defined by

$$\Psi(g)(f) = \int_X g f \, d\mu \quad \forall g \in L^{p'}(X,\mu)$$

is an isometric isomorphism.

Proof:

$$|\Psi(g)(f)| = \left|\int_X g f \, d\mu\right| \le \int_X |g||f| \, d\mu$$

$$\Rightarrow |\Psi(g)(f)| \le \|g\|_{p'} \|f\|_p \quad \forall f \in L^p(X,\mu). \quad (1)$$

$\Psi(g)$ linear functional, bounded $\Rightarrow \Psi(g) \in \left(L^p(X,\mu)\right)^*$

Ψ 1-1 and on-to (see next page). From (1)

$$\|\Psi(g)\| \le \|g\|_{p'} \quad \forall g \in L^{p'}(X,\mu)$$

But

$$\|\Psi(g)\| = \sup\left\{ |\Psi(g)(f)| : f \in L^p, \|f\| = 1 \right\}$$
$$= \sup\left\{ \int_X g f \, d\mu : f \in L^p, \|f\| = 1 \right\}$$
$$= \|g\|_{p'}.$$

$\therefore$ Ψ is isometry.

Thm 165.1 :

Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space. If f is measurable, $1 \le p \le \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\|f\|_p = \sup\left\{ \int fg \, d\mu : \|g\|_{p'} \le 1 \right\}$$

$$\int_X fg \, d\mu \le \|f\|_p \|g\|_{p'} \le \|f\|_p. \quad 1 \le p \le \infty.$$

$$\therefore \sup\left\{ \int_X fg \, d\mu : \|g\|_{p'} \le 1 \right\} \le \|f\|_p.$$

Idea: Construct g with $\|g\|_{p'} = 1$ such that $\int_X fg \, d\mu = \|f\|_p$.

Thm 183.1 : If $1 < p < \infty$ and F is a bounded linear functional on $L^p(X)$, then $\exists \, g \in L^{p'}(X)$ s.t :

$$F(f) = \int_X fg \, d\mu \qquad \forall f \in L^p(X).$$

Moreover $\|g\|_{p'} = \|F\|$ and g is unique.

If $p = 1$, the same conclusion holds assuming that μ is σ-finite.

Idea, define signed measure : $\nu(E) = F(\chi_E)$. Prove that $|\nu(E)| \le \|F\| (\mu(E))^{1/p} \Rightarrow \nu \ll \mu$

Radon - Nikodym Thm $\Rightarrow$

$\exists g$ measurable s.t

$$F(\chi_E) = \nu(E) = \int_X \chi_E g \, d\mu \quad \forall E \in \mathcal{M}.$$

Clearly from here:

$$F(f) = \int_X f g \, d\mu \quad \forall f \text{ simple function}$$

Idea:

- Now approximate a function in L^p by simple functions
- Need to show $g \in L^{p'}$

<u>Def</u>: Let $(X, \mathcal{M}, \mu)$ a measure space and ν another measure on X (possibly signed). We say that ν is absolutely continuous with respecto to μ is:

$$\mu(E) = 0 \Rightarrow \nu(E) = 0.$$

<u>Thm</u> (Radon - Nikodym). If $(X, \mathcal{M}, \mu)$ is a σ-finite signed measure on $\mathcal{M}$ and

$$\nu << \mu$$

Then:

$\exists f$ measurable such that either f^+ or f^- is integrable and $\nu(E) = \int_E f \, d\mu \quad \forall E \in \mathcal{M}.$

Ex: For $1 < p < \infty$,

$L^p(X,\mu)$ is reflexive

Need to show:

$\Phi : L^p(X) \to (L^p(X))^{**}$ is on-to

Let $w \in (L^p(X))^{**}$.

$\therefore \quad w \circ \Psi_1 \in (L^{p'}(X,\mu))^*$ (1)

Recall $\Psi_1 : L^{p'}(X) \to (L^p(X))^*$ isometry.

$\Psi_1(g)(f) = \int gf \, d\mu \quad \forall f \in L^p$

Also: $\Psi_2 : L^p(X) \to (L^{p'}(X))^*$ isometry

$\Psi_2(f)\,(g) = \int fg \, d\mu \quad \forall g \in L^{p'}$

$\therefore$ From (1): $\exists f \in L^p(X,\mu)$ s.t.

$w \circ \Psi_1(g) = \int_X f g \, d\mu = \Psi_1(g)\,(f) \quad \forall g \in L^{p'}$

Check: $\Phi(f) = w$. Let $\lambda \in (L^p(X,\mu))^*$

$\Phi(f)\,(\lambda) \underset{\text{def}}{=} \lambda(f) = w(\lambda)$.

Ex: $L^1(\Omega,\lambda)$ is not reflexive

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Let $\Omega \subset \mathbb{R}^n$ open set

λ lebesgue measure

Exercise 8.19 $\Rightarrow$ $L^1(\Omega,\lambda)$ is separable

Assume that $L^1(\Omega,\lambda)$ is reflexive

$\therefore$ $L^1(\Omega) \sim (L^1(\Omega))^{**}$

$\therefore$ $(L^1(\Omega))^{**}$ is separable

(*) $\therefore$ $(L^1(\Omega))^{*}$ is separable (Thm 287.2)

Since

$$L^\infty(\Omega) \sim (L^1(\Omega))^*$$

and

$L^\infty(\Omega,\lambda)$ is not separable (Exercise 8.20)

We obtain $(L^1(\Omega))^*$ is not separable, which contradicts (*).