Differentiation of Distributions.

Remark: One of the primary reasons why distributions were created, was to provide a notion of differentiability for functions that are not differentiable in the clasical sense.

Motivation for definition of Derivative of a distribution:

Ex: Let $\Omega = (0,1) \subset \mathbb{R}$
f absolutely continuous on $[0,1]$
$f: \mathbb{R} \to \mathbb{R}$
Let $\varphi \in C_c^\infty(0,1)$.

Integration by parts formula. (Problem 7-23 in textbook):
If f and g are absolutely continuous functions defined on $[a,b]$, then

$$\int_a^b f'g \, d\lambda(x) = f(b)g(b) - f(a)g(a) - \int_a^b fg' \, d\lambda(x)$$

Applying this formula to f absolutely continuous and $\varphi \in C_c^\infty(0,1)$ above we get:

$$(A) \quad \boxed{\int_0^1 f'(x)\,\varphi(x)\,d\lambda(x) = -\int_0^1 f(x)\,\varphi'(x)\,d\lambda(x)}$$

- Associate f with a distribution T:

$$T: \mathcal{D}(0,1) \to \mathbb{R}$$
$$T(\varphi) = \int_0^1 \underline{f(x)}\,\varphi(x)\,d\lambda(x), \quad \forall \varphi \in \mathcal{D}(0,1)$$

The derivate of f, f', can be associated with the distribution:

$$S(\varphi) := \int_0^1 \underline{f'(x)}\,\varphi(x)\,d\lambda(x), \quad \forall \varphi \in \mathcal{D}(0,1)$$

By (A), since

$$S(\varphi) = -\int_0^1 f(x)\,\varphi'(x)\,d\lambda(x),$$

then it is natural to define the derivative of T as the distribution:

$$T': \mathcal{D}(0,1) \to \mathbb{R}, \quad \underline{T'(\varphi)} := S(\varphi) = -\int_{(0,1)} f\varphi'\,d\lambda(x) = -\underline{T(\varphi')}$$

Def: Let T be a distribution of order N defined on an open set $\Omega \subset \mathbb{R}^n$. The partial derivative of T with respect to the i th coordinate direction is defined by:

$$\boxed{\frac{\partial T}{\partial x_i}(\varphi) = -T\left(\frac{\partial \varphi}{\partial x_i}\right)}$$

T is a distribution of order $N+1$, Since, for every K compact:

$$\left| T\left(\frac{\partial \varphi}{\partial x_i}\right)\right| \leq C(K) \| \varphi \|_{K,N+1}$$

is true whenever φ is a test function supported on K, on which

$$|T(\varphi)| \leq C(K) \| \varphi \|_{K,N}.$$

Ex 1: $\Omega = (a,b)$
$f : [a,b] \to \mathbb{R}$ absolutely continuous
T is the distribution corresponding to f; ie:
$T : C_c^\infty(a,b) \to \mathbb{R}$
$T(\varphi) = \int_a^b f \varphi \, d\lambda$, $\varphi \in \mathcal{D}(a,b)$

$\therefore$

$$T'(\varphi) = -T(\varphi') = -\int_a^b f\varphi' d\lambda$$

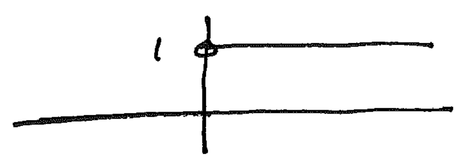
$$= \int_a^b \varphi f' d\lambda \quad ; \quad \varphi(a) = \varphi(b) = 0$$

$$\therefore \; T'(\varphi) = \int_a^b f'\varphi \, d\lambda, \quad \varphi \in \mathcal{D}(a,b).$$

$\therefore$ <u>T' is identified with f'</u>

<u>Ex 2</u>: Let $\Omega = \mathbb{R}$ and define:

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$



Let T be the distribution corresponding to f:

$$T(\varphi) = \int_{\mathbb{R}} f\varphi \, d\lambda, \quad \varphi \in \mathcal{D}(\mathbb{R})$$

$$T'(\varphi) = -T(\varphi') = -\int_{\mathbb{R}} f\varphi' d\lambda$$

$$= -\int_0^\infty \varphi' d\lambda = \varphi(0).$$

$\therefore$ <u>T' is the Dirac measure.</u>

Ex 3: Let $f(x) = |x|$. (10.176)

If T is the distribution associated to f, then:

$$T(\varphi) = \int_{\mathbb{R}} f \varphi \, d\lambda = \int_0^{\infty} x \varphi(x) \, d\lambda(x) - \int_{-\infty}^{0} x \varphi(x) \, d\lambda(x)$$

$$T'(\varphi) = -T(\varphi')$$
$$= -\int_0^{\infty} x \varphi'(x) \, d\lambda(x) + \int_{-\infty}^{0} x \varphi'(x) \, d\lambda(x)$$
$$= \int_0^{\infty} \varphi(x) \, d\lambda(x) - \int_{-\infty}^{0} \varphi(x) \, d\lambda(x)$$

$$\int_0^{\infty} (x\varphi)' = x\varphi \Big]_0^{\infty} = 0$$
$$\int_0^{\infty} (\varphi + x\varphi') = 0 \quad \Rightarrow \quad -\int_0^{\infty} x\varphi' = \int_0^{\infty} \varphi$$

$$= \int_{\mathbb{R}} \varphi(x) g(x) \, d\lambda(x), \quad \text{where } g(x) = \begin{cases} 1, & x > 0 \\ -1, & x \le 0 \end{cases}$$

$$\therefore \boxed{T' = g}$$

Thm: Let T be a distribution in $\mathbb{R}$.

If $T' = 0 \Rightarrow T$ is constant; i.e. T is the distribution that correspond to a constant function.

Proof:

(10.177)

Let $\Upsilon \in \mathcal{D}(\mathbb{R})$ s.t.

$$\int_{\mathbb{R}} \Upsilon(x)\, d\lambda(x) = 1$$

For $\varphi \in \mathcal{D}(\mathbb{R})$, we write:

$$\text{(A)} \quad \boxed{\varphi(x) = [\varphi(x) - a\Upsilon(x)] + a\Upsilon(x)}$$

where:

$$a = \int_{\mathbb{R}} \varphi(t)\, d\lambda(t)$$

Let

$$\alpha := \varphi(x) - a\Upsilon(x)$$

$$\therefore \quad \int_{\mathbb{R}} \alpha(x)\, d\lambda(x) = \int_{\mathbb{R}} [\varphi(x) - a\Upsilon(x)]\, d\lambda(x)$$

$$= a - a \cdot 1 = 0.$$

Note (*) below implies $\exists \beta$ test function with $\beta' = \alpha$.

Since $T'(\varphi) = -T(\varphi') = 0 \quad \forall \varphi$, then:

$$\text{(B)} \quad \boxed{T'(\beta) = -T(\alpha) = 0} \quad ; \quad \text{since } \beta' = \alpha.$$

$\therefore$ From (A) and (B):

$$T(\varphi) = T(\alpha) + a\,T(\Upsilon) = a\,T(\Upsilon)$$

$$= \int_{\mathbb{R}} T(\Upsilon)\, \varphi(x)\, d\lambda(x)$$

Hence, T corresponds to the constant $T(\Upsilon)$.

(*) <u>Note</u>: Let φ be a test function such that $\int_{\mathbb{R}} \varphi(x)\, d\lambda(x) = 0$. Let $\psi(x) := \int_{-\infty}^{x} \varphi(t)\, dt$. Then ψ is a test function with compact support and $\psi' = \varphi$.

Thm : Suppose $f \in L^1_{loc}(a,b)$.
Then

$\exists g$ absolutely continuous, $g = f$ a.e. $x \iff$ The derivative of the distribution corresponding to f is a function

Proof:

$\Rightarrow$ Let $f = g$ almost everywhere where g is absolutely continuous. Let T and S be the distributions corresponding to f and g respectively. We have seen that:

$$S' = g' \text{ ; that is:}$$

$$S'(\varphi) = \int_{\mathbb{R}} g' \varphi \, d\lambda$$

(Recall that, since g is absolutely continuous, then g is differentiable almost everywhere).

We now compute:

$$\begin{aligned} T'(\varphi) &= -T(\varphi') \\ &= -\int_{\mathbb{R}} f \varphi' \, d\lambda(x) \\ &= -\int_{\mathbb{R}} g \varphi' \, d\lambda(x), \quad g = f \text{ almost everywhere} \\ &= \int g' \varphi \, d\lambda(x), \quad \text{integrating by parts.} \\ &= S'(\varphi) \end{aligned}$$

$$\therefore T' = S' = g'$$

$\Leftarrow$ T is the distribution corresponding to f.
Suppose that (10.179)

(A) $\boxed{T' = h}$

Define:

$$g(x) = \int_a^x h(t)\, d\lambda(t)$$

Fundamental Thm of Calculus $\Rightarrow$

- g is absolutely continuous
- $g' = h$, almost everywhere.

Let S be the distribution corresponding to g.

$\Rightarrow$ $S' = g'$; since g is absolutely continuous

$\therefore$ (B) $\boxed{S' = h}$; since $g' = h$

$\therefore$ $(T-S)' = 0$; from (A) and (B)

$\therefore$ $\boxed{T - S = K}$, K is constant.

Thus:

$$\int_{\mathbb{R}} f\varphi\, d\lambda = T(\varphi) = S(\varphi) + \int_{\mathbb{R}} K\varphi = \int g\varphi + K\varphi = \int (g+K)\varphi$$

$\therefore$ $\int_{\mathbb{R}} f\varphi\, d\lambda = \int_{\mathbb{R}} (g+K)\varphi\, d\lambda$, $\forall \varphi \in \mathcal{D}(\mathbb{R})$

$\therefore$ $f = g + K$, almost everywhere (exer. 10.3);

and $g+K$ is absolutely continuous.