Def. (Chapter 7): Let $f:[a,b]\to\mathbb{R}$. The total variation of f from a to x, $x \le b$, is defined by

$$V_f(a;x) = \sup \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})|,$$

where the supremum is taken over all finite sequences:

$$a = t_0 < t_1 < \cdots < t_k = x$$

f is said to be of <u>bounded variation on $[a,b]$</u> if $V_f(a;b) < \infty$. In abbreviated form:

$$f \in BV([a,b]) \quad \text{if} \quad V_f(a;b) < \infty.$$

Def (Chapter 7): Let $f:[a,b]\to\mathbb{R}$. f is said to be <u>absolutely continuous</u> on I (AC on I) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

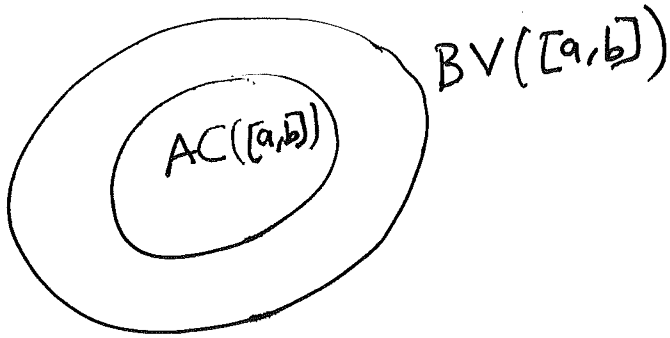
$$\sum_{i=1}^{k} |f(b_i) - f(a_i)| < \varepsilon$$

for any finite collection of nonoverlapping intervals $[a_1,b_1], \ldots, [a_k,b_k]$ in I with:

$$\sum_{i=1}^{k} |b_i - a_i| < \delta.$$

Ex. Any Lipschitz function is absolutely continuous.

In particular, $f(x) = |x|$ is AC on any interval $[a,b]$.



Recall:

- Let $f: \mathbb{R} \to \mathbb{R}$ be a nondecreasing function. Then $f'(x)$ exists λ-a.e. $x \in \mathbb{R}$

- $f: [a,b] \to \mathbb{R}$ satisfies condition N if $E \subset [a,b]$, $\lambda(E) = 0 \Rightarrow \lambda[f(E)] = 0$

- If $f \in AC([a,b]) \Rightarrow f$ satisfies Condition N

- Let $f \in BV([a,b])$. Then:
  $f = f_1 - f_2$, $\quad f_1, f_2$ nondrecreasing
  $\therefore$ f is differentiable almost everywhere

Recall the:

Lebesgue - Stieltjes Measure (Chapter 4).

Let $f: \mathbb{R} \to \mathbb{R}$ non-decreasing

Define:

$$\alpha_f((a,b]) := f(b) - f(a)$$

The Lebesgue-Stieltjes outer measure of an arbitrary set $E \subset \mathbb{R}$ is defined by

$$\lambda_f^*(E) = \inf \left\{ \sum_{h_k \in \mathcal{F}} \alpha_f(h_k) \right\},$$

where the infimum is taken over all countable collections $\mathcal{F}$ of half-open intervals h_k of the form $(a_k, b_k]$ such that

$$E \subset \bigcup_{h_k \in \mathcal{F}} h_k$$

Thm: λ_f^* is a Caratheodory outer measure on $\mathbb{R}$

Thm: If $f: \mathbb{R} \to \mathbb{R}$ is non-decreasing and right continuous, then:

$$\lambda_f((a,b]) = f(b) - f(a)$$

Thm: Let μ be a finite Borel outer measure on $\mathbb{R}$ and let $f(x) := \mu((-\infty, x])$. Then, f is nondecreasing, right continuous and $\lambda_f \equiv \mu$ on all Borel sets.

Thm: Let $f \in L^1_{loc}(a,b)$. Then

$g = f$ λ-almost everywhere, for some $g \in BV[a,b]$ $\iff$ $f' = \mu$, $|\mu|(a,b) < \infty$ (f' is the distributional derivative of f)

Proof: $\Rightarrow$ Assume $f = g$ λ-almost everywhere and $g \in BV([a,b])$. Then

$g = g_1 - g_2$, g_1, g_2 non-decreasing (WLOG assume g_1, g_2 right continuous)

Let T_1 be the distribution corresponding to g_1. Thus:

$$T_1(\varphi) = \int_a^b g_1(x)\varphi(x)\,d\lambda(x)$$

and:

$$T_1'(\varphi) = -T_1(\varphi')$$

$$= -\int_a^b g_1(x)\varphi'(x)\,\underline{d\lambda(x)} \quad ; \text{ Lebesgue integral}$$

$$= -\int_a^b g_1(x)\varphi'(x)\,\underline{dx} \quad ; \text{ Riemann integral.}$$

$$= -\int_a^b g_1(x)\,\underline{d\varphi} \quad ;$$ Riemann-Stieltjes integral; exercises 6.17, 6.18, 6.19 (Ziemer-Torres book).

$$\therefore\; T_1'(\varphi) = -\int_a^b g_1(x)\,d\varphi\,;$$

$d\varphi = \varphi' dx$
Riemann - Stieltjes integral.

$$= \int_a^b \varphi\, dg_1\,;$$

Exercises 6.20
Integration by parts for Riemann - Stieltjes integrals.

$$= \int_a^b \varphi(x) g_1'(x)\,dx\,;$$

$dg_1 = g_1' dx$.
See for example Rudin's book, Chapter 6, Theorem 6.17

$$= \int_a^b \varphi(x)\, g_1'(x)\, d\lambda(x)$$

$$= \int_a^b \varphi(x)\, d\lambda_{g_1}\,;$$

excersice 6.21:
Riemann- Stieltjes and Lebesgue- Stieltjes integrals are in agreement.

(A) $\therefore\; \boxed{T_1'(\varphi) = \int_a^b \varphi\, d\lambda_{g_1}}$

In the same way:

$$\boxed{T_2'(\varphi) = \int_a^b \varphi\, d\lambda_{g_2}, \quad \varphi \in \mathcal{D}(a,b)}$$

3.62 61.2

Hence:

10.185

If T_g is the distribution corresponding to g; since $T_g = T_1 - T_2$ then:

$$\boxed{T_g' = T_1' - T_2' = \lambda_{g_1} - \lambda_{g_2} \text{; by (A)}}$$

Note: If g_1, g_2 are not right continuous, then by Problem 7.10 (see also Theorem 61.2 and Problem 3.62), we can redefine g_1, g_2 to $\tilde{g}_1, \tilde{g}_2$, such that $g_1 = \tilde{g}_1$, $g_2 = \tilde{g}_2$ almost everywhere and:

$\tilde{g}_1, \tilde{g}_2$ are non-decreasing, and right continuous.

Then:

$\tilde{g} := \tilde{g}_1 - \tilde{g}_2$ is of bounded variation (since any non-decreasing function is of bounded variation, $\tilde{g}_1, \tilde{g}_2$ are BV and hence $\tilde{g}$ is BV).

$\therefore$ $f = g = \tilde{g}$ almost everywhere, with $g, \tilde{g} \in BV$.

$$T_f = T_g = T_{\tilde{g}} = \lambda_{g_1} - \lambda_{g_2} = \lambda_{\tilde{g}_1} - \lambda_{\tilde{g}_2}$$

$\Leftarrow$ Suppose now that:

$$f' = \mu \quad ; \quad \text{i.e.}$$

$$-\int_a^b f \varphi' \, d\lambda = \int_a^b \varphi \, d\mu, \quad \forall \varphi \in \mathcal{D}(a,b),$$

where:

μ is a signed measure $\quad |\mu|(a,b) < \infty$

Then:

$$\mu = \mu_1 - \mu_2, \qquad \mu_1, \mu_2 \text{ non-negative measures}$$

Define:

$$f_i(x) = \mu_i((-\infty, x]), \quad i = 1, 2$$

(extend μ_i by $\mu_i(\mathbb{R} \setminus (a,b)) = 0$).

(f_i are non-decreasing and right continuous)

$\therefore$ Theorem 96.1 $\Rightarrow$

$$\mu_i = \lambda_{f_i}, \qquad \lambda_{f_i} \text{ is the Lebesgue-Stieltjes measure}$$

(on all Borel sets)

$$\therefore T'_{f_i}(\varphi) = -T_{f_i}(\varphi') = -\int_a^b f_i \varphi' \, d\lambda = \int_a^b \varphi \, d\lambda_{f_i} = \int_a^b \varphi \, d\mu_i$$

Define

$$\boxed{g := f_1 - f_2}$$

g is BV and $T_g' = T_{f_1}' - T_{f_2}' = \mu_1 - \mu_2 = \mu$

Hence:

$$T_f' = \mu$$

$$T_g' = \mu$$

$$\therefore\ (T_f - T_g)' = 0$$

$$\therefore\ T_f = T_g + K$$

$$\therefore\ \int_a^b f\varphi\, d\lambda = \int_a^b g\varphi\, d\lambda + \int_a^b K\varphi\, d\lambda$$

$$\therefore\ \int_a^b (f-g-K)\varphi\, d\lambda = 0, \quad \forall \in \mathcal{D}(a,b)$$

$\therefore$ $f = g+K$ λ-almost everywhere

Since $g+K$ is BV we conclude the f is equivalent to a BV function.