We now go back to functions of bounded variation in 1- dimension.

The proof of Thm 1 is long (see the detailed proof in Thm 345.2 from Textbook).

The proof uses the equality (since $f' = \mu$) that we proved in Lemma 1:

$$(5) \quad \boxed{\|f'\| = \|\mu\| = \operatorname{Sup}\left\{ \int_a^b \varphi \, d\mu \ : \ \varphi \in C_c^\infty(a,b),\ |\varphi| \le 1 \right\}.}$$

That the distributional divergence of f in the sense of distributions is the finite measure μ is equivalent to:

$$-\int_a^b f \varphi' \, d\lambda(x) = \int_a^b \varphi \, d\mu \, ; \quad \forall \varphi \in C_c^\infty(a,b)$$

Thus, from (5)

$$\boxed{\operatorname{Sup}\left\{ \int_a^b f \varphi' \, d\lambda(x) \ : \ \varphi \in C_c^\infty(a,b),\ |\varphi| \le 1 \right\} < \infty}$$

Alternative definitions of functions of bounded variation (BV) in one-dimension.

Let $f \in L^1(a,b)$. We say that $f \in BV(a,b)$ iff:

(a) $\operatorname{ess} V_a^b f < \infty$

or

(b) The distributional derivative of f is a finite Radon measure in (a,b)

or

(c) $\sup\left\{ \int_a^b f \varphi' \, d\lambda(x) : \varphi \in C_c^1(a,b), |\varphi| \le 1 \right\} < \infty$

We can extend the definition (c) to several variables and define the space of functions of bounded variation in an open set $\Omega \subset \mathbb{R}^n$:

$BV(\Omega)$.

BV functions in $\mathbb{R}^n$.

Def: A function $f \in L^1(\Omega)$, Ω open, has bounded variation in Ω if

$$\sup\left\{\int_\Omega f \operatorname{div} \varphi \, d\lambda(x) : \varphi \in C_c^1(\Omega; \mathbb{R}^n), |\varphi| \leq 1\right\} < \infty.$$

We write

$$BV(\Omega)$$

to denote the space of functions of bounded variation.

Def: A function $f \in L^1_{loc}(\Omega)$ has locally bounded variation in Ω if for each open set $V \subset\subset \Omega$,

$$\sup\left\{\int_V f \operatorname{div} \varphi \, d\lambda(x) : \varphi \in C_c^1(V; \mathbb{R}^n), |\varphi| \leq 1\right\} < \infty.$$

We write

$$BV_{loc}(\Omega)$$

to denote the space of such functions.

Thm: Structure theorem for BV_{loc} functions:

Let $f \in BV_{loc}(\Omega)$. Then there exists a Radon measure μ on Ω and a μ-measurable function $\sigma : \Omega \to \mathbb{R}^n$ such that:

(i) $|\sigma(x)| = 1$ μ-a.e., and

(ii) $\int_\Omega f \operatorname{div} \varphi \, d\lambda(x) = -\int_\Omega \varphi \cdot \sigma \, d\mu$

Moreover, (ii) asserts that the weak first partial derivatives of a BV function are Radon measures.

Proof:

Use the RRT3, local version, to prove (i) and (ii). Then show that (ii) implies that

$$\frac{\partial f}{\partial x_i} = \tilde{\mu}_i, \quad i = 1, \dots, n,$$

where $|\tilde{\mu}_i|(V) < \infty$, for each $V \subset\subset \Omega$.

Complete the details of this proof from Evans' book: "Fine properties of functions..."

Functions of Several Variables (Differentiability)

Let
$f : \mathbb{R} \to \mathbb{R}$.

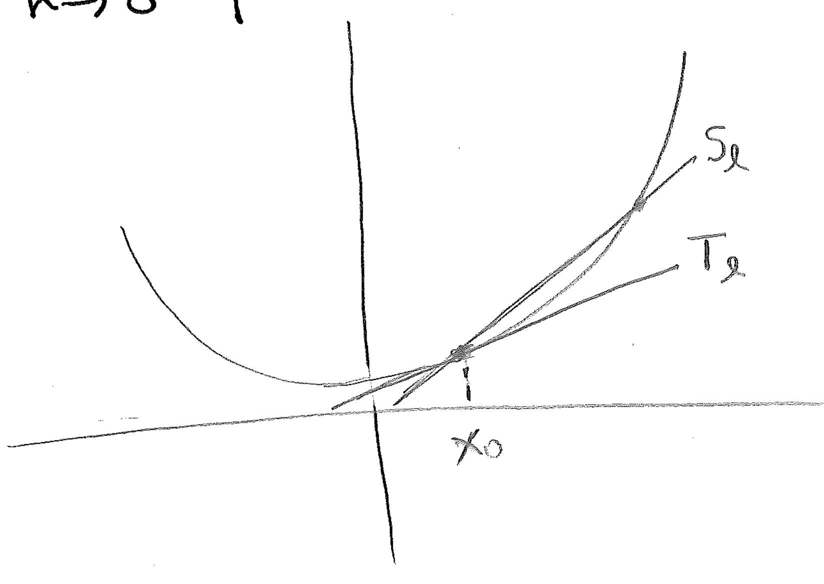
We say that f is differentiable at x_0 if the following limit exists:

$$(1) \quad \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

We denote this limit as $\underline{f'(x_0)}$.

From (1)

$$\lim_{h \to 0} \left| \frac{f(x_0+h) - f(x_0) - f'(x_0) h}{h} \right| = 0$$



Note that the equation of the tangent line T_e is:

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$$y - f(x_0) = f'(x_0)(x - x_0)$$

or

$$T_e(x) = y(x) = f(x_0) + f'(x_0)\,h\,, \qquad h = x - x_0.$$

Thus, $\forall \varepsilon > 0 \;\exists \delta$ s.t $|h| \le \delta \Rightarrow$:

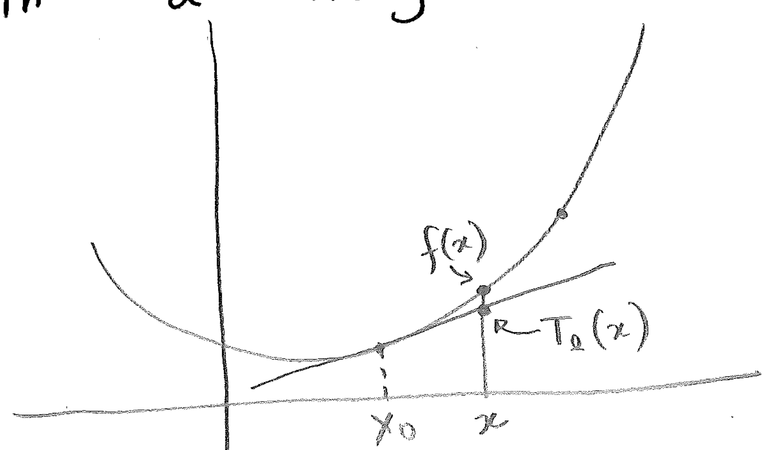
$$|f(x_0 + h) - f(x_0) - f'(x_0)\,h| \le \delta |h|$$

$\therefore$ if $|x - x_0| \le \delta$:

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \le \delta |x - x_0|$$

$$(2) \qquad \boxed{|f(x) - T_e(x)| \le \delta |x - x_0|, \quad |x - x_0| \le \delta.}$$

From (2) we see that the tangent line T_e is the best linear approximation to $f(x)$ in a neighborhood of x_0.



$|x - x_0| < \delta$

We can also write

(3) $\boxed{f(x_0+h) - f(x_0) = f'(x_0)h + r(h)}$

where the "remainder" $r(h)$ is small, in the sense that

$$\lim_{h\to 0} \frac{r(h)}{h} = 0$$

Note that (3) expresses the difference $f(x_0+h)-f(x_0)$ as the sum of the <u>linear function</u> that takes h to $f'(x_0)h$, plus a small remainder.

We can therefore regard the derivative of f at x_0, not as a real number, but as a linear operator of $\mathbb{R}$ that takes h to $f'(x_0)h$.

Observe that every real number α gives rise to the linear operator $L: \mathbb{R} \to \mathbb{R}$, $L(h) = \alpha \cdot h$. Conversely, every linear function that carries $\mathbb{R}$ to $\mathbb{R}$ is multiplication by some real number. It is this natural 1-1 correspondence between $\mathbb{R}$ and $\mathcal{L}(\mathbb{R};\mathbb{R})$ which motivates the preceding statements.

<u>Def</u>: Suppose $E \subset \mathbb{R}^n$, E open, and let $f: E \to \mathbb{R}^m$. If there exits a linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\boxed{\lim_{h \to 0} \frac{|f(x_0+h) - f(x_0) - Lh|}{|h|} = 0,} \quad (4)$$

then we say that f is differentiable at x_0, and we write:

$$df(x_0) = L, \quad L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m).$$

If f is differentiable at every $x \in E$, we say that f is differentiable in E.

Note: $f(x_0+h) - f(x_0) = df(x_0)h + r(h)$, $\frac{|r(h)|}{|h|} \to 0$ as $h \to 0$

Uniqueness of the derivative.

<u>Thm</u>: Suppose E and f are as in previous definition, $x \in E$ and (4) holds with $L = L_1$ and with $L = L_2$. Then $L_1 = L_2$.

<u>Proof</u>: If $B := L_1 - L_2$, the inequality

$$|Bh| \leq |f(x_0+h) - f(x_0) - L_1 h| + |f(x_0+h) - f(x_0) - L_2 h|$$

shows that $\frac{|Bh|}{|h|} \to 0$ as $h \to 0$. For

$h \neq 0$, it follows that:

$$(*) \qquad \frac{|B(th)|}{|th|} \to 0 \quad \text{as } t \to 0$$

The linearity of B shows that the left side of $(*)$ is independent of t. Thus, $Bh = 0$ $\forall h \in \mathbb{R}^n$. Hence $B = 0$.

We have:

<u>Thm</u>: Let $f: E \subset \mathbb{R}^n \to \mathbb{R}^m$, and f is differentiable at $x_0 \in E$. Then the partial derivatives $\frac{\partial f_i}{\partial x_j}(x_0)$ exist, and

$$df(x_0)\, e_j = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(\vec{x}_0)\, u_i ,$$

where $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_m\}$ are the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$.

<u>Proof</u>: Fix e_j. Since f is differentiable at x_0

$$f(x_0 + te_j) - f(x_0) = df(x_0)(te_j) + r(te_j)$$

where $\frac{|r(te_j)|}{t} \to 0$ as $t \to 0$. The linearity of $df(x_0)$ shows therefore that:

$$\lim_{t \to 0} \frac{f(x_0 + te_j) - f(x_0)}{t} = df(x_0)\, e_j$$

$$\therefore \left(\lim_{t \to 0} \frac{f_1(x_0 + te_j) - f_1(x_0)}{t}, \dots, \lim_{t \to 0} \frac{f_m(x_0 + te_j) - f_m(x_0)}{t} \right) = df(x_0)\, e_j ,$$

where we have used that $f=(f_1,\dots,f_m)$.
From here we conclude that $\frac{\partial f_i}{\partial x_j}$, $i=1,\dots,m$ exists.

Noticing that $f=\sum_{i=1}^{m} f_i u_i$ we write:

$$\lim_{t\to 0}\sum_{i=1}^{m}\frac{f_i(x_0+te_j)-f_i(x_0)}{t}u_i = df(x_0)\,e_j$$

$$\therefore \sum_{i=1}^{m}\lim_{t\to 0}\frac{f_i(x_0+te_j)-f_i(x_0)}{t}u_i = df(x_0)\,e_j$$

$$\therefore \sum_{i=1}^{m}\frac{\partial f_i}{\partial x_j}(x_0)\,u_i = df(x_0)\,e_j \qquad \blacksquare$$

Note: From previous Theorem, it is clear that

$$df(x_0)=\begin{pmatrix} \frac{\partial f_1(x_0)}{\partial x_1} & \frac{\partial f_1}{\partial x_2}(x_0) & \dots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \frac{\partial f_2(x_0)}{\partial x_1} & \frac{\partial f_2}{\partial x_2}(x_0) & \dots & \frac{\partial f_2}{\partial x_n}(x_0) \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1}(x_0) & \frac{\partial f_m}{\partial x_2}(x_0) & \dots & \frac{\partial f_m}{\partial x_n}(x_0) \end{pmatrix}$$

In case $m=1$, we have:

$$df(x_0)=\nabla f(x_0)=\left(\frac{\partial f}{\partial x_1}(x_0),\dots,\frac{\partial f}{\partial x_n}(x_0)\right),$$

denoted as the "gradiant of f at x_0".

Ex: Let $f: \mathbb{R}^2 \to \mathbb{R}$

differentiable at (x_0, y_0). Thus:

$$df(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0) \right) = \nabla f(x_0,y_0)$$

We have:

$$\lim_{h \to 0} \frac{| f(x_0+h_1, y_0+h_2) - f(x_0,y_0) - df(x_0,y_0)(h_1,h_2)|}{|h|} = 0$$

where $h_1 = x - x_0$, $h_2 = y - y_0$

Thus:

$$f(x_0+h_1, y_0+h_2) - f(x_0,y_0) = df(x_0,y_0)(h_1,h_2) + r(h_1,h_2)$$

or:

$$f(x,y) - f(x_0,y_0) = \nabla f(x_0,y_0)\,(x-x_0, y-y_0) + r(x-x_0, y-y_0)$$

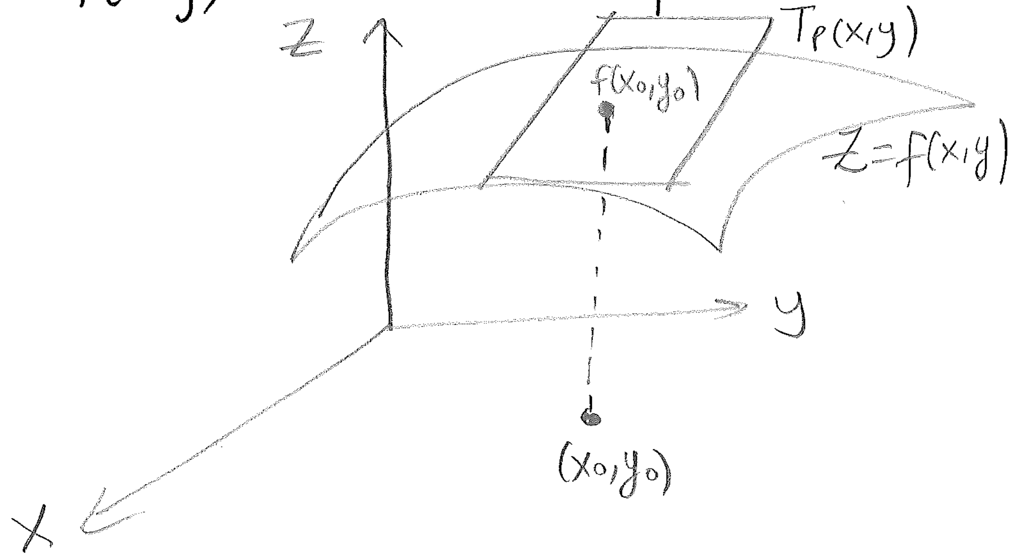
$$f(x,y) = f(x_0,y_0) + \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0) \right) \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + r(x-x_0, y-y_0)$$

$$\therefore \boxed{f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + r(x-x_0, y-y_0)}$$

We recall that

$$T_p(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

is the equation of the tangent plane to the surface $z = f(x,y)$ at the point $(x_0, y_0, f(x_0,y_0))$.



Thus, if $z = f(x,y)$ is differentiable at (x_0,y_0), the tangent plane T_P at $(x_0,y_0,f(x_0,y_0))$ exists and it is the best linear approximation to the graph $z = f(x,y)$ in a neighborhood of (x_0,y_0).