Lipschitz function.

<u>Def:</u> Let $A \subset \mathbb{R}^n$. A function $f : A \to \mathbb{R}^m$ is called Lipschitz provided that:

$$(*) \quad |f(x) - f(y)| \leq C\,|x - y|,$$

for some constant C and all $x, y \in A$.

The smallest constant such that $(*)$ holds for all x, y is denoted:

$$\mathrm{Lip}\,(f) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|},\ x, y \in A,\ x \neq y \right\}.$$

<u>Def:</u> A function $f : A \to \mathbb{R}^m$ is called <u>locally Lipschitz</u> if for each compact $K \subset A$, $\exists\, C_K$ s.t.:

$$|f(x) - f(y)| \leq C_K\,|x - y|, \quad \forall x, y \in K.$$

Rademacher's Theorem.

We next prove Rademacher's remarkable theorem that a Lipschitz function is deferentiable λ-a.e. This is surprising since the inequality

$$|f(x) - f(y)| \leq \mathrm{Lip}(f)\,|x-y|$$

apparently says nothing about the possibility of locally approximating f by a linear map.

Thm: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable λ_n-a.e.

Proof:

We may assume $m=1$. Since differentiability is a local property, we may as well also suppose f is Lipschitz.

Fix $v \in \mathbb{R}$, $|v|=1$, and define:

$$D_v f(x) \equiv \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

provided this limit exists.

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$D_v f(x)$ is the directional derivative of at x in the direction v.

Claim #1 : $D_v f$ exists for λ-a.e. x.

For each $x \in \mathbb{R}^n$, define:

$$\overline{D_v} f(x) := \limsup_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

$\therefore$

$$\text{(A)} \quad \overline{D}_v f(x) = \lim_{k \to \infty} \sup_{\substack{0 < |t| < \frac{1}{k} \\ t \text{ rational}}} \frac{f(x+tv) - f(x)}{t}$$

In order to understand the previous equality (A), we recall the following definition:

Def: $\liminf\limits_{x \to x_0} f(x) := \lim\limits_{r \to 0} m(r, x_0)$

where

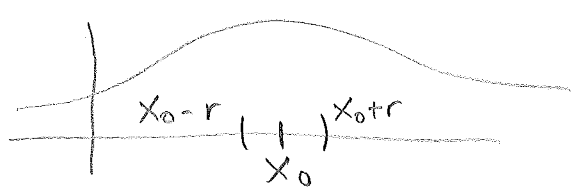
$$m(r, x_0) = \inf \{ f(x) : 0 < |x - x_0| < r \}$$

and

$$\limsup_{x \to x_0} f(x) := \lim_{r \to 0} M(r, x_0),$$

where

$$M(r, x_0) = \sup \{ f(x) : 0 < |x - x_0| < r \}$$



Note: Since rationals are dense, the sup and inf can be taken over all $0 < |x - x_0| < r$, x rational

Claim: The function:

$$x \longmapsto \bar{D}_v f(x)$$

is a Borel measurable map.

Since the rational numbers are countable, we can enumerate all rationals t with $0<|t|<\frac{1}{K}$ as $t_1^K, t_2^K, t_3^K, \ldots$

Define now:

$$G_i^K : \mathbb{R}^n \to \mathbb{R}$$

$$G_i^K(x) = \frac{f(x+t_i^K v)-f(x)}{t_i^K}, \quad i=1,2,\ldots$$

The function G_i^K is Borel measurable since it is continuous (If a function F is continuous then $F^{-1}(B)$ is a Borel set for every Borel set B).

Since the sup of measurable functions is again measurable, it follows that:

$$\sup_i \{G_i^K(x)\} \quad \text{is measurable}$$

Note that:

$$F^K := \sup_i \{G_i^K(x)\} = \sup_{\substack{0<|t|<\frac{1}{K} \\ t \text{ rational}}} \frac{f(x+tv)-f(x)}{t}$$

Thus, for each $K = 1, 2, \ldots$:

F^K is Borel measurable.

Since the pointwise limit of measurable functions is again measurable it follows that:

$$\overline{D}_v f(x) = \lim_{K \to \infty} F^K(x)$$

is Borel measurable.

We now define:

$$\underline{D}_v f(x) := \liminf_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

Proceeding as before we show that

$x \longmapsto \underline{D}_v f(x)$ is Borel measurable.

We have then:

$$\boxed{\begin{array}{l} x \longmapsto \underline{D}_v f(x) \text{ is Borel measurable} \\ x \longmapsto \overline{D}_v f(x) \text{ is Borel measurable.} \end{array}}$$

Define:

$$N_v := \{x \in \mathbb{R}^n : D_v f(x) \text{ does not exist}\}$$
$$= \{x \in \mathbb{R}^n : \underline{D}_v f(x) < \overline{D}_v f(x)\}$$

Since both $\underline{D}_v f(x)$ and $\overline{D}_v f(x)$ are Borel measurable we conclude

$$\boxed{N_v \text{ is Borel}}$$

Claim: For each line L parallel to v we have:

$$\mathcal{H}^1(N_v \cap L) = 0.$$

In order to prove this claim we define:

$$\gamma : \mathbb{R} \to \mathbb{R}$$
$$\gamma(t) = f(x + tv), \quad x, v \in \mathbb{R}^n, |v| = 1$$

Then, γ is Lipschitz, thus absolutely continuous and thus differentiable λ_1-a.e.

Let
$$z : \mathbb{R} \to \mathbb{R}^n$$
$$z(t) = x + tv$$
$$z(0) = x$$

Let $A_v = \{t \in \mathbb{R} : \gamma(t) \text{ is not diffentiable}\}$

$\therefore \lambda_1(A_v) = 0.$

Let $t_0 \notin A_v$. (11.212)

$\therefore$ $\gamma'(t_0)$ exists and

$$\gamma'(t_0) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

$$= \lim_{t \to t_0} \frac{f(x + t v) - f(x + t_0 v)}{t - t_0}$$

(with $h = t - t_0$)

$$= \lim_{h \to 0} \frac{f(x + t_0 v + (t - t_0) v) - f(x + t_0 v)}{h}$$

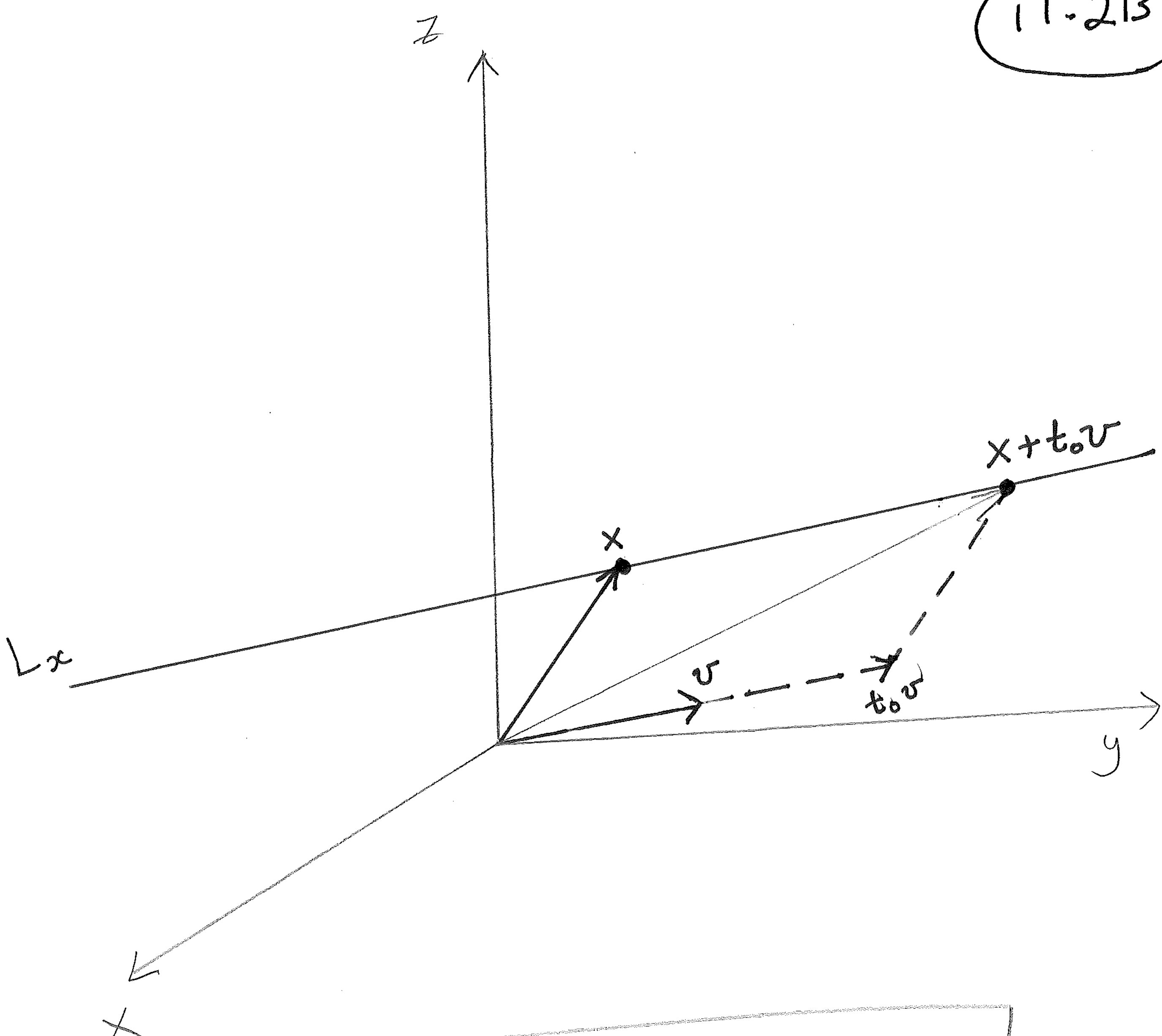
$$= \lim_{h \to 0} \frac{f(x + t_0 v + h v) - f(x + t_0 v)}{h}$$

$$= \lim_{h \to 0} \frac{f(z(t_0) + h v) - f(z(t_0))}{h}$$

$$= D_v f(z(t_0))$$

Hence the directional derivative of f exists, at the point $z(t_0) = x + t_0 v$, in the direction of v.

Note that $z(t_0)$ belongs to L_x, which is the line parallel to v that passes trough x.



We have

$$t_0 \notin A_v \iff z(t_0) = x + t_0 v \notin N_v$$

Let $R_v := z(A_v) \subset L_x$

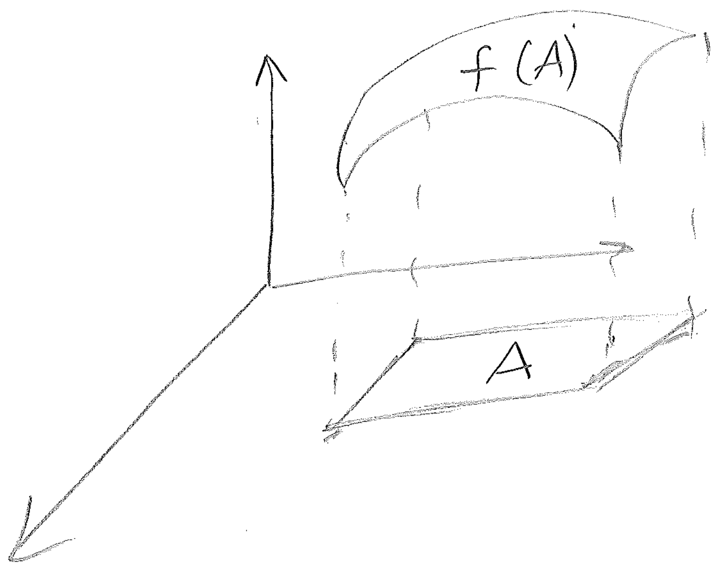
We now claim that

$$\boxed{\mathcal{H}^1(R_v) = 0} \qquad (B)$$

(B) follows from the following theorem:

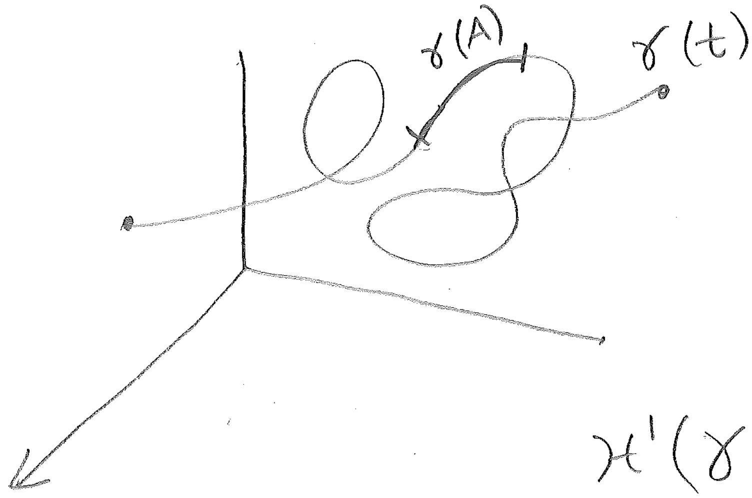
<u>Thm</u>: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $A \subset \mathbb{R}^n$, $0 \le s < \infty$. Then

$$\mathcal{H}^s(f(A)) \le (\mathrm{Lip}(f))^s \, \mathcal{H}^s(A).$$



$f: \mathbb{R}^2 \to \mathbb{R}$ Lipschitz.

$\mathcal{H}^2(f(A)) \le C\, \mathcal{H}^2(A)$.



$\gamma: \mathbb{R} \to \mathbb{R}^3$ Lipschitz

$\mathcal{H}^1(\gamma(A)) \le C\mathcal{H}^1(A)$, $A \subset \mathbb{R}$.

Since $z(t): \mathbb{R} \to \mathbb{R}^n$

$$z(t) = x + tv$$

is Lipschitz, the theorem gives:

$$\mathcal{H}^1(R_v) \leq C\,\mathcal{H}^1(A_v)$$

Since $\mathcal{H}^1(A_v) = \lambda_1(A_v) = 0$, then

$$\mathcal{H}^1(R_v) = 0$$

But

$D_v f(z)$ exists $\iff z \notin R_v$

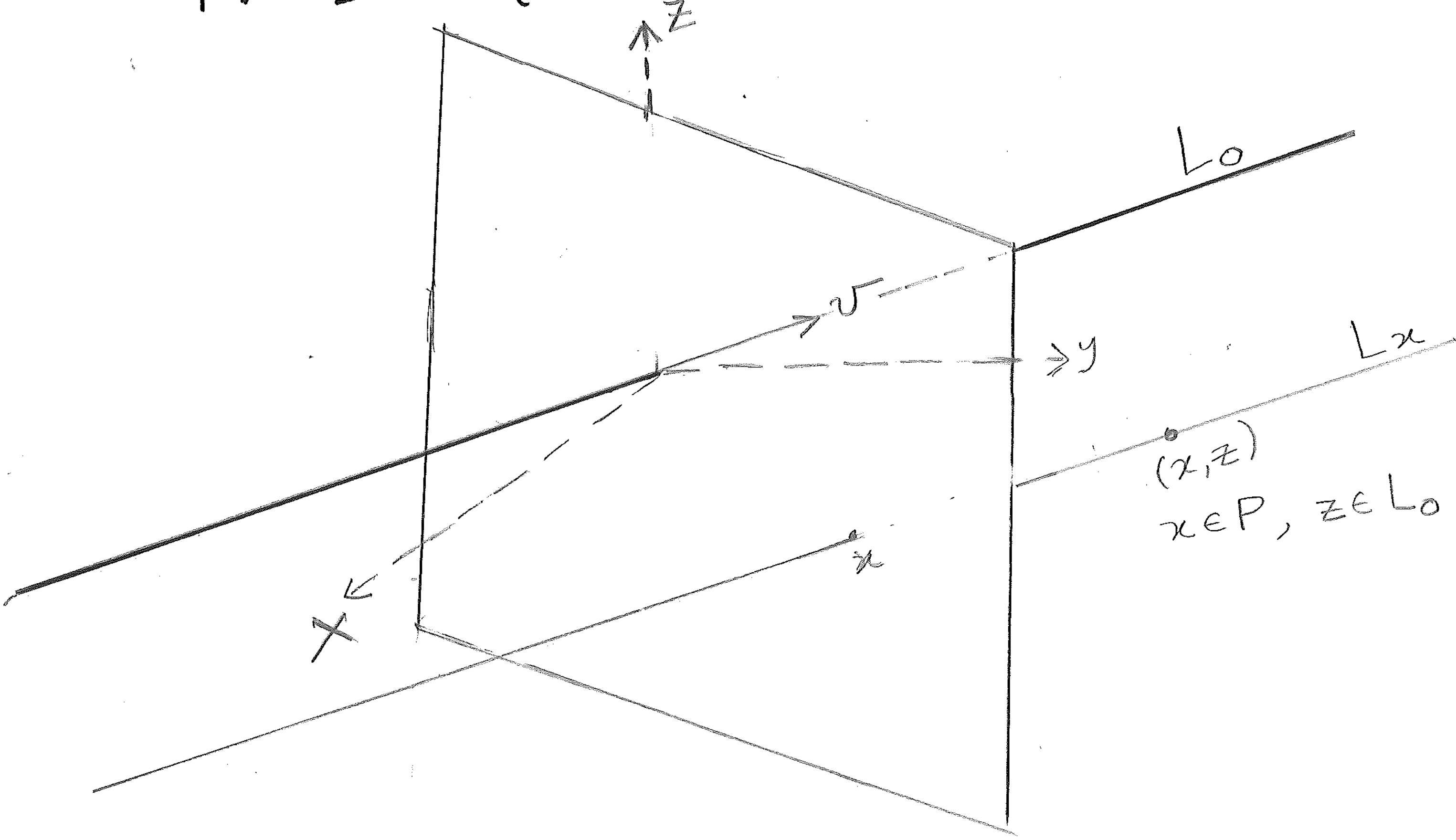
$$\therefore \quad \boxed{\mathcal{H}^1(N_v \cap L_x) = 0}$$

$$\therefore \quad \mathcal{H}^1(N_v \cap L) = 0$$

for every line parallel to v.

Fix now L_0 a line parallel to v. Let:

$$P := L^{\perp} = \{x \in \mathbb{R}^n : \langle x, z \rangle = 0 \ \forall z \in L\}$$



Since N_v is a Borel measurable set, we can apply Fubini's Theorem to
$(\mathbb{R}^n, \mathcal{H}^n) = (P, \mathcal{H}^{n-1}) \times (L_0, \mathcal{H}^1)$

$$\int_{\mathbb{R}^n} \chi_{N_v} \, d\mathcal{H}^n = \int_P \left[\int_{L_0} \chi_{N_v}(x,z) \, d\mathcal{H}^1(z) \right] d\mathcal{H}^{n-1}(x)$$

$$= \int_P \mathcal{H}^1 (N_v \cap L_x) \, d\mathcal{H}^{n-1}(x)$$

$$= 0$$

We conclude the:

$$0 = \int_{\mathbb{R}^n} \chi_{N_v} \, d\mathcal{H}^n = \int_{\mathbb{R}^n} \chi_{N_v} \, d\lambda(x)$$

$$= \lambda_n(N_v)$$

$$\therefore \lambda_n(N_v) = 0$$

Since

$D_v f(z)$ exists $\Longleftrightarrow z \notin N_v$,

and $\lambda_n(N_v) = 0$ we conclude that Claim #1 is true; i.e:

$D_v f$ exists for λ_n-a.e. z.

As a consequence of Claim #1, we see

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right)$$

exists for λ_n-a.e. x.