Claim #2: $D_v f(x) = \nabla f \cdot v$ for λ-a.e. x.

Let $\varphi \in C_c^{\infty}(\mathbb{R}^n)$. Then the change of variables formula yield:

$$\int_{\mathbb{R}^n} \left[\frac{f(x+tv)-f(x)}{t}\right] \varphi(x)\, d\lambda(x) =$$

$$= \int_{\mathbb{R}^n} \frac{f(x+tv)\, \varphi(x)}{t}\, d\lambda(x) - \int_{\mathbb{R}^n} \frac{f(x)\varphi(x)}{t}\, d\lambda(x)$$

$z = x + tv$
$dz = dx$

$$= \int_{\mathbb{R}^n} \frac{f(z)\, \varphi(z-tv)\, d\lambda(z)}{t} - \int_{\mathbb{R}^n} \frac{f(x)\varphi(x)}{t}\, d\lambda(x)$$

$$= \int_{\mathbb{R}^n} \frac{f(x)\, \varphi(x-tv)\, d\lambda(x)}{t} - \frac{1}{t}\int_{\mathbb{R}^n} f(x)\varphi(x)\, d\lambda(x)$$

$$= \int_{\mathbb{R}^n} \frac{(\varphi(x-tv) - \varphi(x))}{t} f(x)\, d\lambda(x), \quad (**)$$

<u>Note:</u> We will study change of variable formulas in the next section.

Let $t = \frac{1}{K}$, $K = 1, \ldots$ in the previous inequality and note:

$$(*) \quad \left| \frac{f(x + \frac{1}{K} v) - f(x)}{\frac{1}{K}} \right| \leq \mathrm{Lip}(f) |v| = \mathrm{Lip}(f)$$

Thus, from $(*)$ and the Dominated Convergence Theorem:

$$\int_{\mathbb{R}^n} D_v f(x) \varphi(x) \, d\lambda(x) = \int_{\mathbb{R}^n} \lim_{K \to \infty} \frac{f(x + \frac{1}{K} v) - f(x)}{\frac{1}{K}} \varphi(x) \, d\lambda(x)$$

$$= \lim_{K \to \infty} \int_{\mathbb{R}^n} \frac{f(x + \frac{1}{K} v) - f(x)}{\frac{1}{K}} \varphi(x) \, d\lambda(x) \text{; Dominated Convergence Theorem}$$

$$= \lim_{K \to \infty} \int_{\mathbb{R}^n} \frac{\left(\varphi(x - \frac{1}{K} v) - \varphi(x)\right)}{\frac{1}{K}} f(x) \, d\lambda(x) \text{; by } (**)$$

$$= -\int_{\mathbb{R}^n} \lim_{K \to \infty} \frac{\varphi(x - \frac{1}{K} v) - \varphi(x)}{-\frac{1}{K}} f(x) \, d\lambda(x)$$

$$= -\int_{\mathbb{R}^n} D_v \varphi(x) f(x) \, d\lambda(x)$$

We have shown:

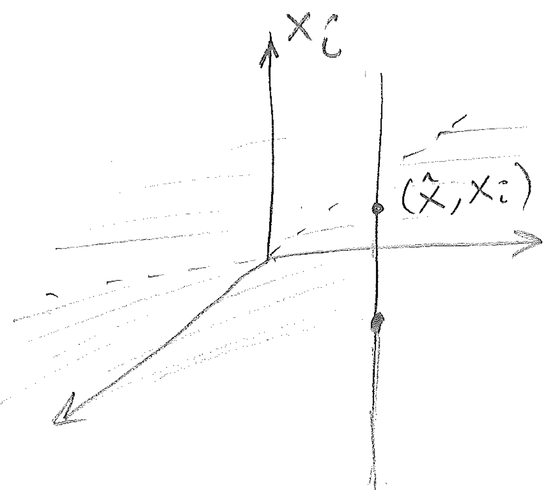
$$\int_{\mathbb{R}^n} D_v f(x)\varphi(x)\,d\lambda(x) = -\int_{\mathbb{R}^n} D_v\varphi(x) f(x)\,d\lambda(x)$$

$$= -\int_{\mathbb{R}^n} f(x)\,\nabla\varphi(x)\cdot v\,d\lambda(x)$$

$$= -\sum_{i=1}^{n} v_i \int_{\mathbb{R}^n} f(x)\frac{\partial\varphi}{\partial x_i}(x)\,dx$$

Using Fubini's Theorem:

$$\int_{\mathbb{R}^n} f(x)\frac{\partial\varphi}{\partial x_i}(x)\,dx = \int_{\mathbb{R}^{n-1}}\left[\int_{\mathbb{R}} f(\hat{x},x_i)\frac{\partial\varphi}{\partial x_i}(\hat{x},x_i)\,d\lambda(x_i)\right] dx_1\ldots dx_{i-1}\,dx_{i+1}\ldots dx_n$$



$x=(\hat{x},x_i)$

$\hat{x}=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$

We can integrate by parts in the inner integral since f is absolutely continuous on lines:

$$\int_{\mathbb{R}^n} f(x)\frac{\partial\varphi}{\partial x_i}(x)\,dx = -\int_{\mathbb{R}^{n-1}}\left[\int_{\mathbb{R}}\frac{\partial f}{\partial x_i}(\hat{x},x_i)\varphi(\hat{x},x_i)\,d\lambda(x_i)\right]d\hat{x}$$

$$= -\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_i}(x)\varphi(x)\,d\lambda(x)$$

Hence:

$$\int_{\mathbb{R}^n} D_v f(x)\, \varphi(x)\, d\lambda(x) = \sum_{i=1}^{n} v_i \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x)\, \varphi(x)\, d\lambda(x)$$

$$= \int_{\mathbb{R}^n} \nabla f(x) \cdot v \quad \varphi(x)\, d\lambda(x)$$

Thus:

$$\int_{\mathbb{R}^n} D_v f(x)\, \varphi(x) d\lambda(x) = \int_{\mathbb{R}^n} \nabla f(x) \cdot v \quad \varphi(x)\, d\lambda(x),$$

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n)$$

Hence

$$D_v f(x) = \nabla f(x) \cdot v, \quad \lambda\text{-a.e. } x$$

Now choose $\{v_k\}_{k=1}^{\infty}$ be a countable dense subset of $\partial B(0,1)$.

Observe that there is a set E, $\lambda(E) = 0$ s.t.:

$$D_{v_k}(f)(x) = \nabla f(x) \cdot v_k,$$

$\forall x \in \mathbb{R}^n \setminus E$.

Claim #3: f is differentiable at each point $x \in \mathbb{R}^n \setminus E$

Let $x \in \mathbb{R}^n \setminus E$, $v \in \partial B(0,1)$. Define: (11.222)

$$Q(x,v,t) := \frac{f(x+tv)-f(x)}{t} - \nabla f(x) \cdot v, \quad t \quad 0$$

For $v, v' \in \partial B(0,1)$ note that:

$$|Q(x,v,t) - Q(x,v',t)| \leq \frac{|f(x+tv) - f(x+tv')|}{|t|} + |\nabla f(x) \cdot (v - v')|$$

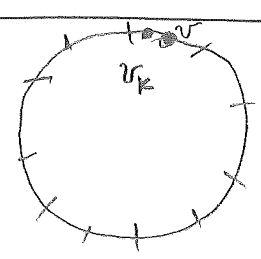
$$\leq \mathrm{Lip}(f)\, C(n)\, |v - v'|$$

$\therefore$

(A) $$\boxed{|Q(x,v,t) - Q(x,v',t)| \leq \mathrm{Lip}(f)\, C(n)\, |v - v'|, \quad v, v' \in \partial B(0,1)}$$

Since $\{v_k\}$ is dense in $\partial B(0,1)$, for every $\varepsilon > 0$, there exists N sufficiently large such that:

(B) $|v - v_k| < \dfrac{\varepsilon}{2C(n)\mathrm{Lip}(f)}$, for every $v \in \partial B(0,1)$ and some k, that depends on v, and $k \in \{1, \dots, N\}$



Thus, for every $v \in \partial B(0,1)$, there exists $k \in \{1, \dots, N\}$ such that:

$$\text{(C)} \quad \boxed{\begin{aligned} |Q(x,v,t)| \le{} & |Q(x,v_k,t)| \\ & + |Q(x,v,t) - Q(x,v_k,t)| \end{aligned}}$$

Since $D_{v_k} f(x) = \underline{\nabla f(x) \cdot v_k}$, then for $0 < |t| \le \delta$ we have $|Q(x, v_k, t)| \le \frac{\varepsilon}{2}$. Thus, from (A), (B), (C):

$$|Q(x,v,t)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad 0 \le |t| \le \delta$$

This shows:

$$\lim_{t \to 0} |Q(x,v,t)| = 0$$

$$\therefore \text{(D)} \quad \boxed{\lim_{t \to 0} \left| \frac{f(x+tv) - f(x)}{t} - \nabla f(x) \cdot v \right| = 0}$$

Now choose any $y \in \mathbb{R}^n$, $y \neq x$. Let

$$v \equiv \frac{y-x}{\|y-x\|}$$

$$\therefore \quad y = x + tv, \qquad t = |y-x|$$

From (D):

$$\lim_{|y-x| \to 0} \frac{|f(y) - f(x) - \nabla f(x) \cdot (y-x)|}{|y-x|} = 0$$

or with $h := y - x$

$$\boxed{\lim_{|h| \to 0} \frac{|f(x+h) - f(x) - \nabla f(x) \cdot h|}{|h|} = 0}$$

and this means the f is differentiable at x. Since $x \notin E$ we conclude

f is differentiable λ-a.e. ∎