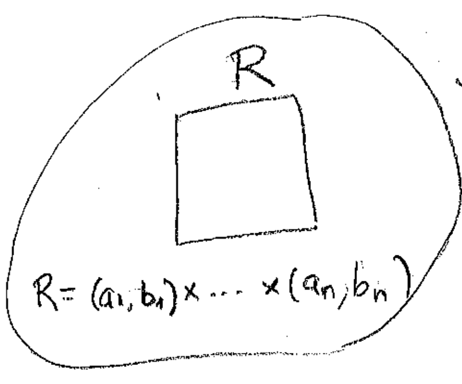
<u>Thm</u>: Suppose $f \in W^{1,p}(\Omega)$, $1 \le p < \infty$. Let $R \subset\subset \Omega$. Then f has a representative f^* that is absolutely continuous on almost all line segments of R that are parallel to the coordinate axes, and the classical partial derivatives of f^* agree almost everywhere with the distributional derivatives of f. Conversely, if f has such a representative and $f \in L^p(R)$, then $f \in W^{1,p}(R)$.

<u>Proof</u>:

Let $f \in W^{1,p}(\Omega)$

Fix i, $1 \le i \le n$.



$\exists \varepsilon_0$ such that:

$$f_\varepsilon = f * \rho_\varepsilon(x)$$

are defined for all $x \in R$, $\varepsilon \le \varepsilon_0$

$$\|f_\varepsilon - f\|_{1,p;R} \to 0 \quad \text{as } \varepsilon \to 0.$$

Let $\varepsilon_K \to 0$ and define:

$$f_K := f_{\varepsilon_K}$$

Let $x \in R$, write:

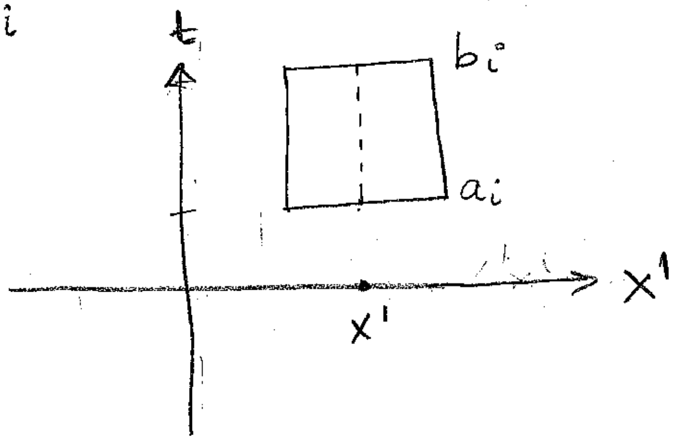
$$x = (x', t)$$

$$x' \in R_i := (a_1, b_1) \times \dots \times \underbrace{(a_i, b_i)}_{\text{omitted}} \times \dots \times (a_n, b_n)$$

$$t \in (a_i, b_i), \qquad 1 \le i \le n.$$

Then:

$$\lim_{K \to \infty} \int_{R_i} \int_{a_i}^{b_i} |f_K(x',t) - f(x',t)|^p + |\nabla f_K(x',t) - \nabla f(x',t)|^p \, dt \, d\lambda(x') = 0$$



$\therefore$

$$\lim_{K \to \infty} \int_{R_i} F_K(x') \, d\lambda(x') = 0 ; \qquad \text{with:}$$

$$F_K(x') = \int_{a_i}^{b_i} |f_K(x',t) - f(x',t)|^p + |\nabla f_K(x',t) - \nabla f(x',t)|^p \, dt$$

$\therefore$ $\boxed{F_K \to 0 \quad \text{in } L^1(R_i)}$

By Vitali's Convergence Theorem (Theorem 168.1), there exits a subsequence of $\{F_K\}$, denoted again as F_K, such that:

$$\boxed{F_K(x') \to 0 \quad \text{for } \lambda\text{-a.e. } x'.}$$

That is; for λ_{n-1}-a.e. $x' \in R_i$:

$$\text{(A)} \quad \boxed{\lim_{K\to\infty} \int_{a_i}^{b_i} |f_K(x',t) - f(x',t)|^p + |\nabla f_K(x',t) - \nabla f(x',t)|^p \, dt = 0}$$

From Hölder's inequality:

$$\text{(B)} \quad \boxed{\int_{a_i}^{b_i} |\nabla f_K(x',t) - |\nabla f(x',t)| \, dt \le (b_i - a_i)^{1/p'} \left(\int_{a_i}^{b_i} |\nabla f_K(x',t) - \nabla f(x',t)|^p \, dt \right)^{1/p}}$$

The Fundamental Theorem of Calculus yields for all $[a,b] \subset [a_i, b_i]$

$$|f_K(x',b) - f_K(x',a)| = \left| \int_a^b \frac{\partial f_K}{\partial x_i}(x',t)\, dt \right|$$
$$\le \int_a^b \left| \frac{\partial f_K}{\partial x_i}(x',t) \right| dt$$
$$\le \int_a^b |\nabla f_K(x',t)| \, dt \quad .$$

Thus

$$\boxed{|f_K(x',b) - f_K(x',a)| \le \int_a^b |\nabla f_K(x',t) - \nabla f(x',t)| \, dt + \int_a^b |\nabla f(x',t)| \, dt .} \quad \text{(C)}$$

Claim: $\{f_K(x',t)\}$ are pointwise bounded: 11.252

Let $\varepsilon = 1$

From (A), (B) and (C), $\exists M(x')$ such that

$$|f_K(x',b) - f_K(x',a)| \le 1 + \int_{a_i}^{b_i} |\nabla f(x',t)|\,dt,$$
$$\forall K > M(x')$$
$$\forall a, b$$
$$[a,b] \subset [a_i, b_i]$$

Note that $\exists$ another subsequence of $\{f_K(x',\cdot)\}$, denoted again as $\{f_K(x',\cdot)\}$ such that

$$f_K(x',t) \to f(x',t), \quad \lambda\text{-a.e. } t.$$

Fix t_0 for which $f_K(x',t_0) \to f_K(x',t_0)$, $t_0 < b$

Then: (with this further subsequence):

$$|f_K(x',b)| \le 1 + \int_{a_i}^{b_i} |\nabla f(x',t)|\,dt + |f_K(x',t_0)|$$

Thus, for K large enough (depending on x'):

$$\boxed{|f_K(x',b)| \le 1 + \int_{a_i}^{b_i} |\nabla f(x',t)|\,dt + |f(x',t_0)| + 1, \quad (D)}$$

which proves the claim.

<u>Claim</u>: $\{f_K(x',\cdot)\}$ is absolutely continuous on $[a_i, b_i]$, for every $k = 1, 2, \ldots$

We have shown earlier in the textbook that:

$f \in AC[a,b] \iff$ (1) f is continuous
(2) $f \in BV[a,b]$
(3) f satisfies Condition N.

Clearly, each $f_K(x',\cdot)$ satisfies these 3 conditions.

Another way to see this is by noticing that, since $f_K(x',\cdot)$ is smooth, we have:

$$f_k(x',t) = \int_{a_i}^{t} \frac{\partial f_k}{\partial x_i}(x',s)\,ds + f_k(x',a_i)$$

Then, the Fundamental Theorem of Calculus yields that $f_K(x',\cdot) \in AC[a_i, b_i]$.

Claim: The functions $\{f_k(x',\cdot)\}$ are absolutely continuous on $[a_i, b_i]$ uniformly on K.

Indeed, given $\varepsilon > 0$, $\exists \tilde{\delta} > 0$ s.t.:

$$\sum |b_I - a_I| < \delta \iff \int_{\cup [a_I, b_I]} |\nabla f(x', t)| \, dt < \varepsilon$$

$[a_I, b_I] \subset [a_i, b_i]$ nonoverlapping intervals

From (C):

$$\sum |f_k(x', b_I) - f_k(x', a_I)| \leq \int_{\cup [a_I, b_I]} |\nabla f_k(x', t) - \nabla f(x', t)| \, dt + \int_{\cup [a_I, b_I]} |\nabla f(x', t)| \, dt$$

$$\leq \varepsilon + \varepsilon, \quad \forall k > M(x')$$

For each $k \in \{1, \dots, M(x')\}$, $\forall \varepsilon > 0$, $\exists \delta_k$ s.t.:

$$\sum |b_I - a_I| < \delta_k \implies \int_{\cup [a_I, b_I]} |\nabla f_k| \, dt < \varepsilon$$

Define $\delta = \min \{\tilde{\delta}, \delta_1, \dots \delta_{M(x')}\}$

Hence:

$$\boxed{\sum |b_I - a_I| < \delta \implies \sum |f_k(x', b_I) - f_k(x', a_I)| \leq 2\varepsilon}$$

for all $k = 1, 2, \dots$ Thus: (E)

$\boxed{f_k(x', \cdot) \text{ is } AC[a_i, b_i] \text{ uniformly on } K \text{ (E)}}$

From (A):

$\{f_K(x',t)\}$ converges for λ_1-a.e. $t \in [a_i, b_i]$ to $f(x',t)$; i.e.

$$\boxed{(F) \quad f_K(x',t) \to f(x',t), \quad \lambda_1\text{-a.e. } t \in [a_i,b_i]}$$

Note:

(a) From (D), the sequence $\{f_K(x',t)\}$ is pointwise bounded on $[a_i, b_i]$

(b) From (E), the sequence $\{f_K(x',t)\}$ is equicontinuous on $[a_i, b_i]$.

Thus, (a)+(b) and Arzela-Ascoli Theorem there is a subsequence of $\{f_K(x',\cdot)\}$ that converges uniformly to a function on $[a_i, b_i]$, say $f_i^*(x',\cdot)$.

Letting $K \to \infty$ in (E) we obtain:

$$f_i^*(x',\cdot) \in AC[a_i, b_i]$$

Also; from (F):

$$\boxed{f(x',t) = f_i^*(x',t) \text{ - for } \lambda\text{-a.e. } t}$$

We conclude:

For each $1 \le i \le n$:

$\exists f_i^*(x',t)$ s.t $f_i^*(x',\cdot) \in AC[a_i,b_i]$ for λ_{n-1} - a.e. $x' \in R_i$; and

$$\boxed{f_i^* = f \text{ almost everywhere on } R}$$

The representative f_i^* was obtained as the pointwise a.e. limit of a subsequence of mollifiers of f.

Note that by iterating n times taking further subsequences in each step $i = 1, \dots, n$, we obtain a sequence of mollifiers, denoted by $\{f_k\}$, and a function f^* such that:

$$\boxed{\begin{array}{l} \text{For every } 1 \le i \le n,\ f_k(x',\cdot) \to f^*(x',\cdot) \text{ uniformly,} \\ \text{for } \lambda_{n-1}\text{ - a.e. } x', \quad f^*(x',t) \text{ is absolutely} \\ \text{continuous in } t \text{ for } \lambda_{n-1}\text{ - a.e. } x', \text{ and} \\ \qquad f^* = f \ \lambda_n\text{-almost everywhere on } R \end{array}}$$

Claim: $\frac{\partial f^*}{\partial x_i}(x)$ exists in the classical sense, for λ_n-a.e. $x \in R$ and:

$$\frac{\partial f^*}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x), \quad \lambda_n\text{-a.e. } x \in R,$$

where $\frac{\partial f}{\partial x_i} \in L^p$ is the distributional divergence of $f \in W^{1,p}(\Omega)$.

We have; for any $\varphi \in C_c^\infty(R)$:

$$\int_R f^* \frac{\partial \varphi}{\partial x_i}\, dx = \int_{R_i} \int_{a_i}^{b_i} f^* \frac{\partial \varphi}{\partial x_i}\, dx_i\, dx'$$

$$= -\int_{R_i} \int_{a_i}^{b_i} \varphi \frac{\partial f^*}{\partial x_i}\, dx_i\, dx'$$

$$= -\int_R \frac{\partial f^*}{\partial x_i} \varphi\, dx$$

Since $f = f^*$ λ_n-a.e x:

$$\int_R f^* \frac{\partial \varphi}{\partial x_i}\, dx = \int_R f \frac{\partial \varphi}{\partial x_i}\, dx$$

$$\therefore \int_R f \frac{\partial \varphi}{\partial x_i} = -\int_R \frac{\partial f^*}{\partial x_i} \varphi, \quad \forall \varphi \in C_c^\infty(R)$$

$$\therefore \int \frac{\partial f}{\partial x_i} \varphi = \int_R \frac{\partial f^*}{\partial x_i} \varphi, \quad \forall \varphi \in C_c^\infty(R)$$

$\Rightarrow$ $\partial f / \partial x_i = \partial f^* / \partial x_i$ almost everywhere