In the following discussion consider:

$$\Omega = \mathbb{R}^n$$

We proved last class that

$$\boxed{\begin{array}{l} C^\infty(\Omega) \cap W^{1,p}(\Omega) \\ \text{is dense in } W^{1,p}(\Omega). \end{array}} \quad (A)$$

For $\underline{\underline{\Omega = \mathbb{R}^n}}$ we also have:

<u>Corollary</u>: If $1 \le p < \infty$, the space $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$

<u>Proof:</u> We can show that

$$\boxed{(B) \quad C_c^\infty(\mathbb{R}^n) \text{ is dense in } C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n),}$$

(with respect to the sobolev norm).

The Corollary follows from (A) and (B).

Let now $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying $\lambda_n(\partial\Omega)=0$

Sobolev functions are only defined almost everywhere.

What does it mean for a Sobolev function to be zero on $\partial\Omega$?

<u>Def</u>: Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set. The space $W_0^{1,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ relative to the Sobolev norm. Thus, $f \in W_0^{1,p}(\Omega)$ if and only if there is a sequence of functions $f_k \in C_c^\infty(\Omega)$ such that:

$$\lim_{k\to\infty} \|f_k - f\|_{1,p;\Omega} = 0$$

Remark 1: If $\Omega = \mathbb{R}^n$, then $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$, since $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$.

Remark 2 : If $\Omega \subsetneq \mathbb{R}^n$ is a bounded open set, then

$$W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$$

and we now proceed to characterize the space $W_0^{1,p}(\Omega)$.

Trace Theorem : Assume Ω is a bounded open set and $\partial\Omega$ is C^1. Then there exists a bounded linear operator:

$$T: W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that:

(i) $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$

(ii) $\|Tu\|_{L^p(\partial\Omega)} \le C \|u\|_{W^{1,p}(\Omega)}$,

for each $u \in W^{1,p}(\Omega)$, with the constant C depending only on p and $\partial\Omega$

Note: $C(\bar{\Omega}) = \{ f : \Omega \to \mathbb{R} \mid f \text{ is uniformly continuous} \}$

If $f \in C(\bar{\Omega})$, then f continuously extends to $\bar{\Omega}$.

Trace-zero functions in $W^{1,p}$:

Assume Ω is bounded and $\partial\Omega$ is C^1. Suppose $u \in W_0^{1,p}(\Omega)$. Then

$$u \in W_0^{1,p}(\Omega) \iff Tu = 0 \text{ on } \partial\Omega$$

We will later prove one of the most useful estimates in the theory of Sobolev functions, which is the Sobolev Imbedding Theorem:

Sobolev Imbedding Theorem: Let $1 \le p < n$ and let Ω be a bounded open subset of $\mathbb{R}^n$. Suppose $f \in W_0^{1,p}(\Omega)$. Then we have the estimate:

$$\|f\|_{q;\Omega} \le C \, \|\nabla f\|_{p,\Omega},$$

for each $q \in [1, p^*]$, $p^* = \dfrac{np}{n-p}$.

The constant C depends only on p, q, n and Ω.

A useful characterization of $W^{1,p}$

Recall that if $f \in L^p(\mathbb{R}^n)$ then:

$$\|f(x+h) - f(x)\|_p \to 0 \quad \text{as } h \to 0$$

We can prove a similar result for Sobolev functions.

<u>Theorem</u>: Let $1 < p < \infty$ and suppose $\Omega \subset \mathbb{R}^n$ is open. If $f \in W^{1,p}(\Omega)$ and $\Omega' \subset\subset \Omega$, the $\dfrac{\|f(x+h) - f(x)\|_{p;\Omega'}}{|h|}$ remains bounded for all sufficiently small h.

Conversely, if $f \in L^p(\Omega)$ and

$$\frac{\|f(x+h) - f(x)\|_{p;\Omega'}}{|h|}$$

remains bounded for all sufficiently small h, then $f \in W^{1,p}(\Omega')$.

Proof:

Let $f \in W^{1,p}(\Omega)$

Let $\Omega' \subset\subset \Omega$.

$\Rightarrow \exists \{f_k\}$, $f_k \in C^\infty(\Omega)$ such that:

$$\| f_k - f \|_{1,p;\Omega} \to 0 \quad \text{as } k \to \infty.$$

For $g \in C^\infty(\Omega)$, we have:

$$\frac{g(x+h) - g(x)}{|h|} = \frac{1}{|h|} \int_0^{|h|} \nabla g\left(x + t\frac{h}{|h|}\right) \cdot \frac{h}{|h|} \, dt$$

$$= \frac{1}{|h|} \int_0^{|h|} \frac{d}{dt} \, g\left(x + t\frac{h}{|h|}\right) dt$$

By Jensen's inequality:

$$\left| \frac{g(x+h) - g(x)}{h} \right|^p \leq \frac{1}{|h|} \int_0^{|h|} \left| \nabla g\left(x + t\frac{h}{|h|}\right) \right|^p dt,$$

for $x \in \Omega'$, $h < \delta := \text{dist}(\partial\Omega', \partial\Omega)$.

Therefore,

$$\| g(x+h) - g(x) \|_{p;\Omega'}^p \leq \frac{|h|^p}{|h|} \int_0^{|h|} \int_{\Omega'} \left| \nabla g\left(x + t\frac{h}{|h|}\right) \right|^p dx \, dt$$

$$\leq |h|^{p-1} \int_0^{|h|} \int_{\Omega} |\nabla g(x)|^p \, dx \, dt$$

$$= |h|^p \int_{\Omega} |\nabla g|^p \, dx$$

$\Rightarrow$

$$\|g(x+h)-g(x)\|_{p;\Omega'} \le |h| \, \|\nabla g\|_{p;\Omega},$$

for $|h|<\delta$.

Hence, we have proved:

$$\frac{\|f_k(x+h)-f_k(x)\|_{p;\Omega'}}{|h|} \le \|\nabla f_k\|_{p;\Omega}$$
$$\le C, \quad |h|<\delta$$
(Since $\|\nabla f_k - \nabla f\|_{p;\Omega} \to 0$)

Letting $k\to\infty$:

$$\frac{\|f(x+h)-f(x)\|_{p;\Omega'}}{|h|} \le C, \quad |h|<\delta.$$

We now prove the converse:

Let $e_i = \{0, 0, \ldots, 1, 0 \ldots 0\}$

By assumption:

$$\frac{\|f(x+\frac{e_i}{k})-f(x)\|_{p;\Omega'}}{1/k} \le C, \quad \text{large } k$$

$\therefore$ $\exists$ a subsequence of $\left\{\frac{f(x+e_i/k)-f(x)}{1/k}\right\}_{k=1}^{\infty}$ (denoted by the full sequence) and $f_i \in L^p(\Omega')$ such that:

$$\frac{f(x+e_i/h)-f(x)}{1/k} \longrightarrow f_i \text{ weakly in } L^p(\Omega').$$

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Let $\varphi \in C_0^\infty(\Omega')$

$\therefore$

$$\int_{\Omega'} f_i \varphi \, d\lambda(x) = \lim_{k \to \infty} \int_{\Omega'} \left[\frac{f(x + e_i/k) - f(x)}{1/k} \right] \varphi(x) \, dx$$

$$= \lim_{k \to \infty} \int_{\Omega'} f(x) \left[\frac{\varphi(x - e_i/k) - \varphi(x)}{1/k} \right] dx$$

$$= - \int_{\Omega} f(x) \frac{\partial \varphi}{\partial x_i} \, dx$$; by Dominated Convergence Theorem.

$\therefore$ $\frac{\partial f}{\partial x_i} = f_i$, in the sense of distributions

$\therefore$ $f \in W^{1,p}(\Omega')$.