Laplace's equation

Problem: Find a function $f \in C^2(\Omega)$ such that:

$$\frac{\partial^2 f(x)}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}(x) + \cdots + \frac{\partial^2 f(x)}{\partial x_n^2} = 0$$

for each $x \in \Omega$ and

"$f(x) = \psi(x)$" on $\partial\Omega$,

(in a weak sense).

Definition: A function $f \in C^2(\Omega)$ is called harmonic in Ω if

$$\boxed{(A) \quad \frac{\partial^2 f}{\partial x_1^2}(x) + \cdots + \frac{\partial^2 f}{\partial x_n^2}(x) = 0,}$$

for each $x \in \Omega$.

(A) is equivalent to:

$$\boxed{(B) \quad \operatorname{div}(\nabla f)(x) = 0, \quad x \in \Omega.}$$

Definition: We say that f is harmonic in the sense of distributions or weakly harmonic if:

$$\int_\Omega f \, \Delta\varphi \, dx = 0, \quad \varphi \in C_c^\infty(\Omega)$$

Note: Let $f \in C^2(\Omega)$. Then

f is harmonic in Ω $\Longleftrightarrow$ f is weakly harmonic in Ω

$\Longrightarrow$ Let f harmonic in Ω

$\therefore \Delta f = 0$ in Ω

$\therefore \varphi \Delta f = 0, \quad \varphi \in C_c^\infty(\Omega)$

$\therefore \int_\Omega \varphi \Delta f = 0$

$\therefore \int_\Omega \varphi \operatorname{div}(\nabla f) \, dx = 0$

Now:

$$\int_\Omega \operatorname{div}(\varphi \nabla f) = \int_\Omega \varphi \operatorname{div}(\nabla f) + \int_\Omega \nabla\varphi \cdot \nabla f \, dx$$

$$\int_{\partial\Omega} \varphi \nabla f \cdot \nu \, dx^{n-1} = 0$$

Hence:

(C) $\int_\Omega \varphi \Delta f \, dx = -\int_\Omega \nabla\varphi \cdot \nabla f \, dx, \quad \forall \varphi \in C_c^\infty(\Omega)$

Interchanging f and φ in previous equality we have:

(D) $\int_\Omega f \Delta \varphi \, dx = -\int_\Omega \nabla\varphi \cdot \nabla f \, dx, \quad \forall \varphi \in C_c^\infty(\Omega)$

$\therefore \int_\Omega f \Delta \varphi \, dx = \int_\Omega \varphi \Delta f \, dx = 0$

$\therefore \int_\Omega f \Delta \varphi \, dx = 0, \quad \forall \varphi \in C_c^\infty(\Omega)$

$\therefore$ f is weakly harmonic

$\Leftarrow$ Suppose

$\int_\Omega f \Delta \varphi \, dx = 0, \quad \forall \varphi \in C_c^\infty(\Omega)$

From (C) and (D):

$\int_\Omega f \Delta \varphi \, dx = \int_\Omega \varphi \Delta f \, dx = 0$

(E) $\therefore \int_\Omega \varphi \Delta f \, dx = 0, \quad \forall \varphi \in C_c^\infty(\Omega)$

Since (E) is zero for any test function φ we conclude $\Delta f(x) = 0$. Hence f is harmonic.

Note: Keep in mind that it is always possible to define ΔT, whenever T is a distribution. In particular if T is represented by the function f then:

$$\Delta f : C_c^\infty(\Omega) \to \mathbb{R}$$

is the distribution given by:

$$\Delta f(\varphi) = \int_\Omega f \Delta \varphi \, dx .$$

Lemma: Let $f \in W^{1,2}(\Omega)$. Then

f is weakly harmonic in Ω $\iff$ $\int_\Omega \nabla f \cdot \nabla \varphi \, dx = 0$, for all $\varphi \in C_c^\infty(\Omega)$

Proof: Exercise 11.10 for Homework.

We will prove in the next 3 classes:

Theorem 1: Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open set & let $\psi \in W^{1,2}(\Omega)$. Then, there exists a weakly harmonic function $f \in W^{1,2}(\Omega)$ such that $f - \psi \in W_0^{1,2}(\Omega)$

Theorem 2: A weakly harmonic function $f \in W^{1,2}(\Omega)$ is of class $C^\infty(\Omega)$.