

11.283

## Laplace's equation

Problem: Find a function  $f \in C^2(\Omega)$

such that:

$$\frac{\partial^2 f(x)}{\partial x_1^2} + \frac{\partial^2 f(x)}{\partial x_2^2} + \dots + \frac{\partial^2 f(x)}{\partial x_n^2} = 0$$

for each  $x \in \Omega$  and

$$"f(x) = \psi(x)" \text{ on } \partial\Omega,$$

(in a weak sense).

Definition: A function  $f \in C^2(\Omega)$  is called harmonic in  $\Omega$  if

$$(A) \quad \frac{\partial^2 f}{\partial x_1^2}(x) + \dots + \frac{\partial^2 f}{\partial x_n^2}(x) = 0,$$

for each  $x \in \Omega$ .

(A) is equivalent to:

$$(B) \quad \operatorname{div}(\nabla f)(x) = 0, \quad x \in \Omega.$$

Definition: We say that  $f$  is harmonic in the sense of distributions or weakly harmonic if:

$$\int_{\Omega} f \Delta \varphi \, dx = 0, \quad \varphi \in C_c^{\infty}(\Omega)$$

Note: Let  $f \in C^2(\Omega)$ . Then

$f$  is harmonic in  $\Omega \iff f$  is weakly harmonic in  $\Omega$

$\implies$  Let  $f$  harmonic in  $\Omega$

$$\therefore \Delta f = 0 \text{ in } \Omega$$

$$\therefore \varphi \Delta f = 0, \quad \varphi \in C_c^{\infty}(\Omega)$$

$$\therefore \int_{\Omega} \varphi \Delta f = 0$$

$$\therefore \int_{\Omega} \varphi \operatorname{div}(\nabla f) \, dx = 0$$

Now:

$$\int_{\Omega} \operatorname{div}(\varphi \nabla f) = \int_{\Omega} \varphi \operatorname{div}(\nabla f) + \int_{\Omega} \nabla \varphi \cdot \nabla f \, dx$$

$$\int_{\Omega} \varphi \nabla f \cdot \nu \, dx^{n-1} = 0$$

Hence:

$$(C) \int_{\Omega} \psi \Delta f \, dx = - \int_{\Omega} \nabla \psi \cdot \nabla f \, dx, \quad \forall \psi \in C_c^\infty(\Omega)$$

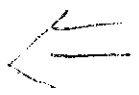
Interchanging  $f$  and  $\psi$  in previous equality we have:

$$(D) \int_{\Omega} f \Delta \psi \, dx = - \int_{\Omega} \nabla \psi \cdot \nabla f \, dx, \quad \forall \psi \in C_c^\infty(\Omega)$$

$$\therefore \int_{\Omega} f \Delta \psi \, dx = \int_{\Omega} \psi \Delta f \, dx = 0$$

$$\therefore \int_{\Omega} f \Delta \psi \, dx = 0, \quad \forall \psi \in C_c^\infty(\Omega)$$

$\therefore f$  is weakly harmonic



Suppose

$$\int_{\Omega} f \Delta \psi \, dx = 0, \quad \forall \psi \in C_c^\infty(\Omega)$$

From (C) and (D):

$$\int_{\Omega} f \Delta \psi \, dx = \int_{\Omega} \psi \Delta f \, dx = 0$$

$$(E) \therefore \int_{\Omega} \psi \Delta f \, dx = 0, \quad \forall \psi \in C_c^\infty(\Omega)$$

Since (E) is zero for any test function  $\psi$  we conclude  $\Delta f(x) = 0$ . Hence  $f$  is harmonic.

Note: Keep in mind that it is always possible to define  $\Delta T$ , whenever  $T$  is a distribution. In particular if  $T$  is represented by the function  $f$  then:

$$\Delta f : C_c^\infty(\Omega) \rightarrow \mathbb{R}$$

is the distribution given by:

$$\Delta f(\varphi) = \int_{\Omega} f \Delta \varphi \, dx.$$

Lemma: Let  $f \in W^{1,2}(\Omega)$ . Then

$f$  is weakly harmonic in  $\Omega$   $\iff \int_{\Omega} \nabla f \cdot \nabla \varphi \, dx = 0$ ,  
for all  $\varphi \in C_c^\infty(\Omega)$

Proof: Exercise 11.10 for Homework.

We will prove in the next 3 classes:

Theorem 1: Suppose  $\Omega \subset \mathbb{R}^n$  is a bounded open set and let  $\gamma \in W^{1,2}(\Omega)$ . Then, there exists a weakly harmonic function  $f \in W^{1,2}(\Omega)$  such that  $f - \gamma \in W_0^{1,2}(\Omega)$

Theorem 2: A weakly harmonic function  $f \in W^{1,2}(\Omega)$  is of class  $C^\infty(\Omega)$ .