Some Differential Equations of Mechanics and Technology.

Isaac Newton said:

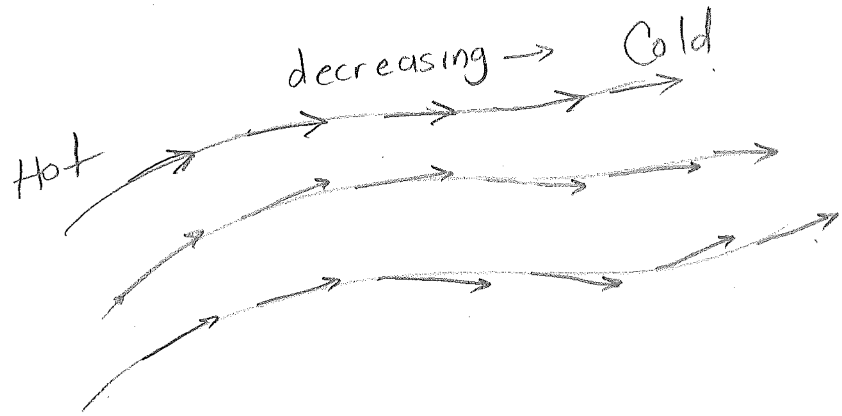
"All in nature reduces to differential equations"

Max Planck said:

"... Present day physics, as far as it is theoretically organized, is completely governed by a system of space-time differential equations".

Conservation of Energy and the Derivation of the Heat equation.

If $T(t, x, y, z)$ (a C^2 function) denotes the temperature in a body at time t, then ∇T represents the temperature gradient, Heat "Flows" with the vector field $-\nabla T = \vec{F}$



$$-\nabla T = -\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)$$

$\nabla T(t_0, x, y, z)$ points in the direction of maximal increasing T.

$$D_u T(x,y,z) = \nabla T(x,y,z) \cdot u$$
$$= \|\nabla T(x,y,z)\| \cos\theta$$

Max attain at $u = \frac{\nabla T}{\|\nabla T\|}$

Min attain at $u = -\frac{\nabla T}{\|\nabla T\|}$

The energy density, that is, the energy per unit volume is (8.41)

$\rho = C\rho_0 T$ where
C = specific heat, constant
ρ_0 = mass density, constant

$\vec{J} = K\vec{F}$ is the energy flux vector
K conductivity constant

Physical truth: "Energy is conserved".
"Mass is conserved".

Physical truth in Math Language:

$$\frac{d}{dt}\iiint_W \rho \, dV = -\iint_{\partial W} \vec{J}\cdot\vec{n}\, dS.$$

Then:

$$\iiint_W \frac{\partial \rho}{\partial t}\, dV = -\iint_{\partial W} \vec{J}\cdot\vec{n}\, dS$$

$$\iiint_W \frac{\partial \rho}{\partial t}\, dV = -\iiint_W \operatorname{div}\vec{J}\, dV \quad (1)$$

(8, 42)

By divergerce Thm:

$$\iiint_W \operatorname{div} \vec{J}\, dV = \iint_{\partial W} \vec{J} \cdot \vec{n}\, dS$$

$\therefore$ From (1):

$$\iiint_W \left(\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{J} \right) dV = 0 \quad (2)$$

Since (2) is true for every region W we obtain:

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{J} = 0} \quad (3)$$

Note:

$$\begin{aligned}
\operatorname{div} \vec{J} &= \operatorname{div}(-K \nabla T) \\
&= K \operatorname{div}\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right) \\
&= -K \left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) \right] \\
&= -K \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \\
&= -K \, \Delta T
\end{aligned}$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(c\rho_0 T) = c\rho_0 \frac{\partial T}{\partial t} \qquad (8.43')$$

Thus (3) becomes:

$$c\rho_0 \frac{\partial T}{\partial t} - K \Delta T = 0 ,$$

or

$$\boxed{\frac{\partial T}{\partial t} = \frac{K}{c\rho_0} \Delta T} \qquad (4) \quad \text{HEAT EQUATION}$$

where $k = \frac{K}{c\rho_0}$ is called the diffusivity

PDE (4) governs heat conduction:
If $T(0, x, y, z)$ is a given initial temperature distribution, then a unique $T(t, x, y, z)$

equation (4).

Remark: If T does not change with time (the steady-state case), then T must satisfy:

$$\boxed{\Delta T = 0} \quad \text{Laplace's equation.}$$